A DETERMINISTIC VORTEX METHOD FOR SOLVING THE NAVIER-STOKES EQUATIONS*

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ABSTRACT: In this paper, a new 2-D vortex method is developed, which treats the vorticity diffusion in a deterministical way. The Laplacian operator, which describes vorticity diffusion, is approximated by a contour integral. The numerical results of two model problems show that this method has a good accuracy. A primary error estimation is given, and the self-adaptive vortex blob and the boundary conditions are discussed.

KEY WORDS: deterministic vortex method, Navier-Stokes equation

I. INTRODUCTION

Vortex method is a Lagrangian method for solving viscous, incompressible fluid flows. The continuous vorticity field is discretized into small vortices. The simulation of fluid flow is implemented by computing the interactions and motions of these vortices. Vortex method has many advantages. Because it is a Lagrangian method, the numerical dissipation is very small, and may be self-adaptive. Especially with high Reynolds numbers, vorticity field is limited in small regions near the solid surfaces, resulting in high resolution of the flowfield with a limited number of vortex elements. Due to these advantages the vortex method has been developing very fast and applied to many flow simulation problems in the past decade.

Among vortex methods, the most mature one is the vortex blob method with random walks put forward by Chorin\cite{3}. Vortex blob method solves Navier-Stokes equation in three steps. In the first step the convective motion of blobs is calculated by solving Euler equation. Since vortex blob has a core, the singularity of a point vortex is avoided, and a good numerical stability is achieved. The vortex blob method simulates vorticity diffusion by random walks of the blobs and the non-slip boundary condition is satisfied by generating the nascent blobs from solid surfaces. This method is simple and easy to implement, and enjoys a very good adaptability for complicated body configurations. Therefore the random vortex method is competitive in solving Navier-Stokes equation with a high Reynolds number. However this method has some shortcomings. First of all, the accuracy of simulating the viscous diffusion by random walks is poor. For unsteady flow problems the random vortex method could not have a good accuracy, not only because it deals only with the statistical average, but also because every random walk may be considered as a small disturbance to the unsteady flow field. For the bounded flow problem, affected by the random process of the vorticity diffusion the distribution of the nascent blobs circulation is also randomized to a great extent.

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The deterministic methods for treating the vorticity diffusion have been developed to overcome the shortcoming of random vortex method, while keeping its advantages. In Choquin and Huberson’s fractional method\cite{2} and Lu’s Lagrangian-Eulerian hybrid method\cite{6}, they introduce the exact similarity solution of a point-vortex diffusion to treat the vorticity diffusion in a deterministic way. Another deterministic way to treat the viscous diffusion is the vorticity rearrangement method (Raviart\cite{9}, Mas-Gallic et al.\cite{7}), that is, the vortex element’s circulation is adjusted according to the vorticity diffusion equation. Fishelov\cite{4} used a different method, which explicitly differentiates the vortex element cut-off function to construct an approximate Laplacian operator.

Differing from the above-mentioned methods, Russo\cite{10} used the vortex elements to construct Voronoi’s polygons, on which the Laplacian operator is discretized. From the numerical testing, Russo found that the accuracy of his method is $O(N^{-1})$. Another deterministic method is the extension of the vortex-in-cell method, which is also a hybrid Lagrangian-Eulerian method. Chang\cite{1} used this method to simulate the impulsively started motion of a cylinder. The new advances in the deterministic vortex methods were reviewed by Tong and Yin\cite{11}.

Now we develop a deterministic method to compute the vorticity diffusion. Starting from the physical background that the vorticity field is a continuous field, we get an approximate differential operator for the viscous diffusion through the geometric relation between neighboring vortex elements; then the approximate differential operator is discretized with the help of the contour integral. Our method is similar to Russo’s in some sense, but the physical meaning and the mesh construction are different. The discrete differential operator in our method is an explicit one, and it can treat the non-slip boundary conditions in a deterministic way and construct the self-adaptive vortex blobs. In comparison with the other deterministic vortex methods, our method provides a unified deterministic way in solving the Euler equation and viscous diffusion, and in treating the boundary conditions.

II. THE DETERMINISTIC VORTEX METHOD

The 2-D non-dimensional Navier-Stokes equations for the incompressible, viscous fluid are given as

\[
\begin{align*}
\frac{\partial \omega}{\partial t} + (V \cdot \nabla)\omega &= \frac{1}{Re} \Delta \omega \\
\nabla \cdot V &= 0
\end{align*}
\]

(2.1)

(2.2)

where $\omega$ is vorticity. In two-dimensional flow along the pathline of a fluid particle we have

\[
\frac{dx}{dt} = V(x, y, t)
\]

(2.3)

\[
\frac{d\omega}{dt} = \frac{1}{Re} \Delta \omega
\]

(2.4)

We solve this problem by the fractional method. The first is the convective step, that is, to solve Eq.(2.3), determine the convective velocity of fluid particles and the new positions of vortex elements at the next time step. The second step is to solve the vorticity diffusion Eq.(2.4).

1. Solving Vorticity Diffusion Equation

Because the vorticity field is a continuous field, the vorticity at any point in the flowfield can be obtained by interpolation from the neighboring vortex elements. First we have to