STEADY GROWTH OF CRACKS IN COMPRESSIBLE ELASTIC-PLASTIC MEDIA *

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ABSTRACT: Based on the theoretical framework for crack growth analysis provided by Gao and Hwang, the 5-sector solution of near-tip fields of mode-I cracks growing quasi-statically and steadily in compressible elastic perfectly plastic materials is obtained. As Poisson’s ratio \( \nu \) tends to \( 1/2 \), the 5-sector solution degenerates to the 4-sector solution of near-tip fields of crack growth in incompressible elastic perfectly plastic materials.

KEY WORDS: compressible materials, elastic perfectly plastic medium, growing crack, near-tip fields.

I. BASIC EQUATIONS AND SUPPLEMENTARY CONDITIONS

Denote by \( x_1, x_2, x_3 \) the fixed rectangular coordinates, with \( x_1 \) along the direction of crack growth. Denote by \( r, \theta \) the plane polar coordinates centered at the crack-tip. For steady crack growth, assuming the rate of crack growth \( d = 1 \), we have

\[
(\dot{\phi}) = -\frac{\partial}{\partial x_1} (\phi) = -\left( \cos \theta \frac{\partial}{\partial r} \rho - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \rho \right)
\]

where \( (\dot{\phi}) \) denotes material derivative with respect to time.

Take the leading term of Airy’s stress function \( \phi(r, \theta, \nu) \) in the form

\[
\phi(r, \theta, \nu) = r^2 F(\theta, \nu)
\]

where \( \nu \) is Poisson’s ratio. Then we have

\[
\sigma_r = F'' + 2F \quad \sigma_\theta = 2F \quad \sigma_{\theta\theta} = -F'
\]

where prime denotes partial derivative with respect to \( \theta \).

The Mises yield condition is

\[
\frac{1}{4} F''^2 + F' \zeta^2 + \frac{1}{3} \varepsilon^2 \zeta^2 = \tau_0^2
\]

where

\[
\varepsilon = \frac{1}{2(1-\nu)} \quad \zeta = -\frac{3}{2} \frac{\varepsilon_{11}}{\varepsilon}
\]

From Hooke’s law and Prandtl-Reuss relation, considering (1), the dominant term of constitutive relation is

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\[
\hat{\varepsilon}_{ij} = \frac{1}{r} \sin \theta \left( \frac{1 + v}{E} \sigma_{ij} - \frac{v}{E} \delta_{ij} \sigma_{rr} \right) + \lambda s_{ij} \quad (i, j, k = 1, 2, 3)
\]

The condition of plane strain, \( \varepsilon_{zz} = 0 \), leads to

\[
\zeta' + \frac{2}{3} \frac{1}{\sin \theta} Er \lambda \zeta = F'' + 4F'
\]

The "pseudo" plastic strain components are\footnote{\cite{11}}

\[
\varepsilon_{ii}^p = \frac{1}{2} \int_{x_1}^{x(A)} \lambda (\sigma_{ii} - \sigma_{22}) dx_1 + \varepsilon^2 \frac{1}{E} (\sigma_{ii} + \sigma_{22} - \zeta) \]

\[
\varepsilon_{22}^p = \frac{1}{2} \int_{x_1}^{x(A)} \lambda (\sigma_{22} - \sigma_{11}) dx_1 + \varepsilon^2 \frac{1}{E} (\sigma_{11} + \sigma_{22} - \zeta) \]

\[
\varepsilon_{12}^p = \int_{x_1}^{x(A)} \lambda \sigma_{12} dx_1
\]

The integration above is along the direction parallel to \( x_i \)-axis, and the upper limit \( x(A) \) denotes the value of \( x_i \) at the boundary of plastic zone ahead of crack-tip.

The compatibility equation and its rate are, respectively,

\[
\frac{1 - v^2}{E} \Delta \Delta \varphi + \frac{\partial^2 \hat{\varepsilon}_{ii}^p}{\partial x_1^2} + \frac{\partial^2 \hat{\varepsilon}_{12}^p}{\partial x_1 \partial x_2} - 2 \frac{\partial^2 \hat{\varepsilon}_{12}^p}{\partial x_1 \partial x_2} = 0
\]

\[
\frac{1 - v^2}{E} \Delta \Delta \varphi + \frac{\partial \varphi}{\partial x_1} - \frac{1}{2} (\nabla_1 \nabla_1 - \nabla_2 \nabla_2) \cdot (\nabla_1 \nabla_1 - \nabla_2 \nabla_2) \lambda - 2 (\nabla_1 \nabla_2) (\nabla_2 \nabla_1 \varphi)
\]

\[
- \frac{1}{2} \lambda \Delta \Delta \varphi - (\nabla_1 \lambda) (\nabla_1 \Delta \varphi) - \frac{2}{3} \varepsilon^2 \Delta (\lambda \zeta) = 0 \quad (\alpha = 1, 2)
\]

It was proved by Gao and Hwang\footnote{\cite{12}} that, the following five contiguity conditions should be satisfied at the bordering lines (as lines of discontinuity) between neighboring sectors;

\[
[F]_r = 0
\]

\[
[F']_r = 0
\]

\[
[F'']_r = 0
\]

\[
\frac{1 - v^2}{E} \frac{1}{r} [F''']_r + \frac{1}{2 \sin \theta_r} [\lambda F'']_r + \frac{2}{3} \varepsilon^2 \frac{1}{\sin \theta_r} [\lambda \zeta]_r - 2 \frac{d}{dr} [\hat{\varepsilon}_{ii}^p]_r = 0
\]

\[
\frac{1 - v^2}{E} \frac{1}{r^2} [F^{(a)}]_r + \frac{1 - v^2}{E} \cot \theta_r \frac{1}{r^2} [F'']_r + \frac{1}{2 \sin \theta_r} \frac{1}{r} [\lambda' F'']_r
\]