INTEGRABILITY FOR PFAFFIAN CONSTRAINED SYSTEMS: A GEOMETRICAL THEORY*

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ABSTRACT: There exists an Ehresmann connection on the fibred constrained sub-manifold defined by Pfaffian differential constraints. It is proved that curvature of the connection is closely related to the $d$-$\delta$ commutation relation in the classical nonholonomic mechanics. It is also proved that conditions of complete integrability for Pfaffian systems in Frobenius sense are equivalent to the three requirements upon the conditional variations in the classical calculus of variations: (1) the variations belong to the constrained manifold, (2) variational operators commute with differential operators, (3) variations satisfy the Chetaev's conditions. Thus this theory verifies the conjecture or experience of researchers of mechanics on the integrability conditions in terms of variation calculus.

KEY WORDS: nonholonomic constraints, Frobenius integrability, fibre bundle, Ehresmann connections, curvature form

1 INTRODUCTION

In many mechanical systems, such as nonholonomic mechanical systems, singular systems, dynamical systems of infinite dimensions, hydrodynamics, etc., there exist differential constraints upon the motion of mechanical systems. According to Frobenius integrability conditions, these constraints can be classified into two groups: holonomic constraints and nonholonomic constraints. A holonomic constraint reduces the phase space in two dimensions, i.e., it reduces the configuration manifold and the fibre space of a tangent bundle in the same dimensions. Consequently, the reduced phase space of the constrained system has symplectic structure as a tangent bundle. However, a nonholonomic constraint does not restrict the configuration manifold and it affects the fibres only, which corresponds to the restriction upon the freedom of the mechanical system. So this submanifold is of co-dimension 1 and does not have symplectic structure of a tangent bundle.

On the other hand, the calculus of variations continues to be used in the study of classical nonholonomic mechanics following holonomic mechanics\textsuperscript{[1]}. But unlike the case
of holonomic mechanics, many realizations of nonholonomic constraints make it impossible for the variations to satisfy the following three requirements simultaneously: (1) Variations belong to constrained submanifold, (2) Variational operators commute with differential operators, (3) Variations obey the Chetaev's conditions. The different selections among them, such as (1) and (2), (2) and (3), or (1) and (3), lead to Vacco variation, Holder's variation and Suslov's variation, respectively. Most researchers believe that these variations are equivalent to each other or the above three requirements can be satisfied simultaneously only if the differential constraints are integrable in the sense of Frobenius theorem\cite{2}. In fact, the non-uniqueness of conditional variations for constrained mechanical systems is closely related to the fact that sub-manifolds subject to nonholonomic constraints do not have symplectic structures.

It should be pointed out that it is only an experience of researchers of mechanics to regard the above three requirements as their criterion of integrability for differential constraints, which seems to lack of sufficient mathematical foundations. Obviously, it is very important to examine or to verify the conclusion in a mathematical point of view. This is our motivation for studying such a problem. In this paper the global differential geometry is used to bridge over the two aspects. There exists an Ehresmann connection defined by the constraints on the constrained sub-manifold\cite{3}. And the further investigation shows that the curvature form of the connection, which represents the non-involution of horizontal distributions on the fibred constrained sub-manifold, corresponds to the noncommutativity of differential operator $d$ and variational operator $\delta$ in calculus of variations. Furthermore, the non-involution of horizontal distributions is equivalent to the non-integrability of the constraints. So the constraints are integrable in Frobenius sense only if the variational operator of Suslov's type is commutative with the differential operator, i.e., the above three conditions upon the variations are satisfied at the same time.

The Einstein's summation convention will be used throughout this paper and $\alpha, \beta = 1, 2, \ldots, g < n; \mu, \sigma = 1, 2, \ldots, \varepsilon = n - g; s = 1, 2, \ldots, n$.

2 **EHRESMANN CONNECTIONS ON FIBRED CONSTRAINED SUBMANIFOLD**

Consider a time-dependent mechanical system with $n$ generalized coordinates. Let $M$ be its smooth extended configuration manifold of dimension $n + 1$ with local coordinates $\{t, q^a\}$, which is a fibred manifold over $\mathbb{R}$ with canonical projection $\pi: M \rightarrow \mathbb{R}$. $TM$ denotes the tangent bundle to $M$ and $J_1M$ the 1-jet bundle. Suppose that the system is subject to $g$ linear nonholonomic constraints

$$q^{\alpha} + \gamma = B^\alpha_{\alpha} (t, q^a) q^\alpha + B^\alpha_{\gamma} (t, q^a) \quad (1)$$

which is equivalent to a Pfaffian system. This construction distinguishes two lots of coordinates, $\{q^\alpha\}$ and $\{q^{\alpha+\beta}\}$. If the coordinate transformations keep this distinction, the configuration manifold $M$ is of fibration structure over a manifold of dimension $n - g + 1$ with local coordinates $\{t, q^a\}$, denoted by $M_0$. In fact, $M_0$ is affine sub-manifold of $M$.

Let $\tau: M \rightarrow M_0$, $\pi_0: M_0 \rightarrow \mathbb{R}$, then $\pi = \pi_0 \cdot \tau$. $J_1M_0$ denotes a 1-jet manifold and it is sub-manifold of $TM_0$. $VM$ denotes the vertical subspace of $TM$ with respect to the projection $\tau$, i.e., the kernel of differential mapping $d\tau$. $\tau^*(J_1M_0)$ represents the pull back