THE DISTRIBUTION OF SOLID PARTICLES SUSPENDED IN A TURBULENT FLOW: A STOCHASTIC APPROACH

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ABSTRACT: Basic fluid mechanics and stochastic theories are applied to show that the concentration distribution of suspended solid particles in a direction normal to the mean streamlines of a two-dimensional turbulent flow is greatly influenced by the lift force exerted on them in the vicinity of the wall. Analytic solution shows that, when the direction of the mean flow is horizontal, the probability density function \( p(y, t) \) for random displacements of the particles will have a maximum value at a point from the wall where the perpendicular component of the lift force precisely balances particle gravity. Interpretation of experimental observations is presented using this theory.

KEY WORDS: suspended particles, turbulent diffusion, random motion, probability density distribution

I. INTRODUCTION

Most of the classical equations for the concentration distribution of solid particles suspended in a turbulent flow were derived from the assumption that the turbulent diffusion of particles could be described using Fick's law of diffusion, and that the diffusion coefficient was proportional to eddy viscosity. These theories are not concerned with the motion of discrete particles and are not to be applied to the near-wall region in two-dimensional flows of dilute particle suspensions, where particles will be subject to some lift effect and dramatic particle concentration reduction has been observed in experimental studies\(^{1,2}\). Such problems should be approached from a point of view of the random motion of particles.

Tchen\(^3\) was the first to attempt to explain the mechanism of turbulent diffusion of solid particles in a way entirely different from the classical ones. He derived an equation for the motion of a solid particle in a homogeneous turbulent field, and developed a differential equation for particle number density function on the basis of the Fokker-Planck-Kolmogorov theory of Brownian motion. The correspondence between these two equations has not been found out, therefore the result of Tchen's theory is not applicable to particle motion near a plane wall if the lift effect is not negligible.

Here we will follow a simpler approach and show that the probability density function describing particle displacements satisfies a Fokker-Planck equation, which can be derived in terms of random motion of discrete particles in a turbulent field, and which can include the influence of lift effect on the motion of a solid particle in the vicinity of a plane wall.

II. DESCRIPTION OF PARTICLE MOTION

Let \( y \) denote the direction normal to the mean streamlines of a two-dimensional turbulent flow. Consider a simple case in which the mean flow is in the horizontal direction, so that \( y \) is the vertical direction. The \( y \)-component of the Newtonian equation of the motion of a particle with mass \( m_p \) and velocity \( v_p \) is

\[
m_p \frac{dv_p}{dt} = f + w_z + F_s
\]

where \( f \), \( w_z \), and \( F_s \) are drag force, gravity of submerged particle and random force on the particle due to fluid turbulence, respectively. Hjelmfelt and Mockros\(^4\) developed the concept of frequency response of a solid particle in an oscillating flow field. Particle velocity \( v_p \) and fluid velocity \( v_f \) are expressed in the following Fourier integrals

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\[ v_f = \int_{0}^{+\infty} (\zeta \cos \omega t + \beta \sin \omega t) d\omega \]  

\[ v_f = \int_{0}^{+\infty} \{ \eta [ \zeta \cos (\omega t + \psi) + \beta \sin (\omega t + \psi) ] \} d\omega \]  

where \( \omega \) denotes frequency, \( \psi \) is the phase angle, \( \zeta \) and \( \beta \) are amplitudes of Fourier components, \( \eta \) is the amplitude ratio. Using Tchen's equation of particle motion\(^3\), Hjelmfelt and Mockros were able to find out the relationship between \( \eta \) and the Stokes number \( N_s = [ \frac{\nu}{(\omega D^2)} ]^{1/2} \), as shown in Fig.1, where \( \nu \) denotes kinematic viscosity of fluid. The calculation was made for \( \nu = 10^{-6} \text{ m/s} \), \( D = 0.06 \sim 2 \text{ mm} \), and \( \rho_s / \rho = 2.65 \), where \( \rho_s \) and \( \rho \) are densities of particle and fluid, respectively. It can be seen that there is a striking difference between spectra of \( v_f \) and \( v_p \) at higher frequencies, since a solid particle could not respond to fluid velocity fluctuations of high frequency due to its inertia. A more straightforward description of this situation is given by Lee and Durst\(^5\), which is shown in Fig. 2, where \( \omega \) denotes the characteristic frequency of eddy motion which is inversely proportional to the characteristic size of eddy motion, \( l_e \). The three ranges of \( \omega \) shown in Fig. 2 (a) correspond to the three ranges of \( l_e \) in Fig. 2 (b). Lee and Durst noted that the motion of a particle is dominated by the eddy motion whose characteristic size is much greater than the size of the particle (i.e., \( l_e >> D \)). In contrast, eddies of much smaller sizes than particle size (i.e., \( l_e << D \)) could hardly have any impact on the motion of the particle.

In this paper, a length scale \( l_e = D \) is chosen to be the size characteristic of the smallest eddy motion to which a particle of diameter \( D \) could respond. From simple dimensional analysis we can obtain its corresponding time scale

\[ t_m \sim \frac{D}{v_*} \]  

where \( v_* = (\tau_w / \rho)^{1/2} \) is shear velocity, \( \tau_w \) is wall shear stress. We then decompose the turbulent velocity fluctuation into \( v_f1 \) and \( v_f2 \), defined by the following Fourier integrals

\[ v_f = v_f1 + v_f2 \]

\[ v_f1 = \int_{0}^{\omega_0} (\zeta \cos \omega t + \beta \sin \omega t) d\omega \]  

\[ v_f2 = \int_{\omega_0}^{+\infty} (\zeta \cos \omega t + \beta \sin \omega t) d\omega \]  

where \( \omega_0 = 2\pi / t_m \) is the cut-off frequency. The fluctuation of particle velocity is determined by \( v_f1 \). 

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**Fig.1** Relationship between \( \eta \) and \( N_s \)

**Fig.2** Particle-eddy interaction (from [5])