ON THE STABILITY OF AN INVARIANT TORUS

Let $\varphi^0 \in \Omega_\varphi$. Then we have a sequence $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{n \to \infty} t_n = +\infty \quad \text{and} \quad \lim_{n \to \infty} \varphi(t_n, \varphi) = \varphi^0.$$ 

Consequently,

$$\lim_{n \to \infty} \varphi(t + t_n, \varphi) = \lim_{n \to \infty} \varphi(t, \varphi(t_n, \varphi)) = \varphi_t(\varphi^0) \subset \Omega_\varphi.$$ 

Thus, condition (3) is equivalent to the condition

$$\limsup_{t \to +\infty} \left| \frac{\text{Re}(\lambda_j(P(\varphi_0)))}{\text{Re}(\lambda_j(P(\varphi)))} \right| < 0, \quad \forall \varphi \in \mathcal{S}_m, \quad j = 1, \ldots, n. \quad (4)$$

Moreover, one can indicate time $T > 0$ such that, for all $t \geq T$, condition (4) is satisfied for any $\varphi \in \mathcal{S}_m$.

(i) Denote

$$-\beta = \max_{j, \varphi \in \Omega} \text{Re}(\lambda_j(P(\varphi))) < 0, \quad -\gamma = -\beta + \varepsilon_1,$$

where $\varepsilon_1 > 0$ is chosen so that $-\gamma < 0$. Then the following estimate (see, e.g., [1]) is true:

$$\|e^{P(\varphi)(t-T)}\| \leq D_1e^{-\gamma(t-T)} \equiv \eta(t-T),$$

$$D_1 = D_1(\varepsilon_1) = \text{const} > 0.$$

We choose $\lambda$ so that the condition $-\gamma < -\lambda < 0$ is satisfied. Then condition (i) of Theorem 1 is satisfied in the form

$$\|e^{P(\varphi)(t-T)}\| = D_1e^{-\gamma(t-T)} \leq D_1e^{-\lambda(t-T)}.$$ 

(ii) Consider condition (ii) of Theorem 1. It is known that the matrix $P(\varphi)$, $\varphi \in \mathcal{S}_m$, satisfies the Lipschitz condition with respect to $\varphi \in \mathcal{S}_m$, i.e.,

$$\|P(\varphi) - P(\psi)\| \leq k\|\varphi - \psi\|,$$

where $\varphi, \psi \in \mathcal{S}_m$ and $k > 0$. Since $\phi = \varepsilon_1(\varphi)$, we have
where

\[ K_1 = \max_{\varphi \in \mathcal{S}_m} \|a(\varphi)\|, \]

i.e., condition (ii) of Theorem 1 is satisfied with the function

\[ \alpha(t-s) \equiv \varepsilon K_1 |t-s|. \]

(iii) Consider the condition

\[ \int_0^{+\infty} e^{\lambda z} \alpha(z) \eta(z) dz < 1. \]  

(5)

In the case under consideration, this condition takes the form

\[ \varepsilon K_1 K_1 \int_0^{+\infty} z e^{(\lambda-\gamma)z} dz < 1. \]

Since \( \lambda - \gamma < 0 \), we have

\[ \int_0^{+\infty} z e^{(\lambda-\gamma)z} dz = \frac{1}{\lambda - \gamma} e^{(\lambda-\gamma)z} \bigg|_0^{+\infty} - \frac{1}{\lambda - \gamma} \int_0^{+\infty} e^{(\lambda-\gamma)z} dz = \frac{1}{\gamma - \lambda} \int_0^{+\infty} e^{(\lambda-\gamma)z} dz \]

\[ = \frac{1}{\gamma - \lambda} \left[ \frac{1}{\lambda - \gamma} e^{(\lambda-\gamma)z} \bigg|_0^{+\infty} \right] = \frac{1}{(\gamma - \lambda)^2}. \]

By substituting the obtained result in (5), we get

\[ \frac{\varepsilon K_1 K_1}{(\gamma - \lambda)^2} < 1 \Rightarrow \varepsilon K_1 K_1 < (\gamma - \lambda)^2 \Rightarrow -\gamma + \sqrt{\varepsilon K_1 K_1} < -\lambda < 0. \]

The last inequality implies the following condition for the existence of the parameter \(-\lambda < 0\):

\[ -\gamma + \sqrt{\varepsilon K_1 K_1} < 0 \Rightarrow \varepsilon K_1 K_1 < \gamma^2 \Rightarrow \varepsilon < \frac{\gamma^2}{K_1 K_1}. \]

Corollary. In order that the conditions of Theorem 1 be satisfied and the matriciant of system (2), beginning with a certain \( T > 0 \), satisfy the estimate

\[ \|\Omega_T'(\varphi)\| \leq D_1 e^{-\lambda (t-T)}, \]