ON THE CONSTRUCTION OF CDN[]-GROUPS WITH ELEMENTARY COMMUTANT OF RANK TWO

M. M. Semko

We describe certain CDN[-groups of order \( p^n \) with elementary commutant of rank two.

CDN[-groups are a natural and rather significant generalization of Dedekind groups. The study of CDN[-groups was begun in [1] and continued in [2–5]. Locally graded meta-Hamiltonian and nonnilpotent groups of such a kind were described in these papers. It is easy to show that the description of arbitrary nilpotent CDN[-groups can be obtained from the description of CDN[-groups of order \( p^n \). In the present paper, we describe certain CDN[-groups \( G \) of order \( p^n \) with elementary commutant of rank two (Theorems 1 and 2). Necessary definitions can be found, e.g., in [5].

We essentially use the following lemmas in the proof of main results:

**Lemma 1.** All CDN[-groups \( G \) which contain the dihedral subgroup of order 8 are meta-Hamiltonian groups.

**Lemma 2.** A nilpotent nonmeta-Hamiltonian CDN[-group \( G \) which contains the nonnormal subgroup of quaternions \( Q = \langle x, y \rangle \) of order 8 is either the group of quaternions of order 32 or a finite nonmetacyclic 2-group with commutant \( G' \) of order \( 2^m \), \( m > 1 \), that possesses the normal subgroup \( C = Q \times \langle d \rangle, \ |d| = 2, \Phi(Q) \times \langle d \rangle < G, \Phi(Q) \times \langle d \rangle \) contains all involutions from \( G \), and \( G/C \) is a Dedekind group.

**Lemma 3.** Let \( G \) be a nonmeta-Hamiltonian CDN[-group \( G \) of order \( p^n \) which contains an elementary Abelian subgroup \( M \) of order \( p^3 \). Then \( |G'| = p^m, \ m > 1, \ n > 4, \) all elements of order \( p \) from \( G \) belong to \( M < G, \ G/M \) is a Dedekind group, and the following statements are true:

(i) \( G \) contains the nonnormal subgroup

\[
X = \langle x \rangle \lambda \langle y \rangle, \ |x| = p^\Delta, \ |y| = p, \ \Delta > 1, \ [x, y] = c = x^{p^\Delta-1} \in Z(G),
\]

\[
[X: \langle c \rangle] > 4, \ \langle x^p \rangle \times \langle y \rangle < G, \ \langle c \rangle \times \langle y \rangle = w(x) < G,
\]

\( G \) does not contain non-Abelian subgroups of order \( p^3 \) which are 2-groups or have the exponent \( p \);

(ii) for any elements \( u \) and \( v \) from \( G, \ |u| > p, \ |v| > p, \) and \( \langle u, v \rangle = \langle u \rangle \lambda \langle v \rangle, \) we have \( w((u, v)) = w((u)) \times (v) < G \) in \( G; \) Frattini subgroups of all cyclic subgroups and commutants of all non-Abelian subgroups are normal;

(iii) \( G \) in the 2-group \( G' \leq Z(\Phi(G)) \).

**Proof.** Assume that \( G \) and \( M \) satisfy the condition of the lemma. Then \( G \) is a nonmetacyclic group and \( n > 4, \ |G'| = p^m, \ m > 1. \) In view of Lemmas 1 and 2, \( G \) does not have dihedral subgroups of order 8 and all...
subgroups of quaternions of order 8 are normal in $G$. By virtue of Theorem 3.4 from [3], $M < G$ and $G/M$ is a Dedekind group. By virtue of Theorem 2.2 from [3], in the nonmeta-Hamiltonian group $G$, there exists a non-normal subgroup $X = \langle x \rangle \cup \langle y \rangle$ of the type (v) of the mentioned theorem. Therefore,

$$|x| = p^\alpha, \quad |y| = p, \quad \Delta > 1, \quad [x, y] = c = x^{p^{\alpha-1}} \in Z(G),$$

$$[X : \langle c \rangle] > 4, \quad \langle x^p \rangle \times \langle y \rangle < G, \quad \langle c \rangle \times \langle y \rangle = w(X) < G.$$

Let $C = C_G(w(X))$. Then $C < G$ and $C < G$. As is known [6], $G/C$, which is a subgroup of $\text{Aut} w(X) = GL(2, p)$, and the Sylow $p$-subgroup of $GL(2, p)$ have order $p$. Consequently, $[G : C] = p$. It is clear that $x \in C$ and, therefore, $G = C(x) = Cx, \quad C \cap X = \langle x^p \rangle \times \langle y \rangle < G$. We set $U = MX$. Obviously, $U < G$ and there exists a series $X < U_1 \leq U$, where $[U_1 : X] = p, \quad [U : U_1] \leq p$. Since $X$ is a metacyclic group and $M$ is a nonmetacyclic subgroup, $M \cap X$ contains $z, \quad |z| = p$. We can assume that $U_1 = X \times \langle z \rangle$. In view of Theorem 2.2 from [3], $U_1 < G$. It is easy to show that any element $d$ from $G$ of order $p$ belongs to $M$. Consequently, $G$ does not contain non-Abelian subgroups of order $p^3$ and the exponent $p, M = w(X) \times \langle z \rangle, U_1 = U$. Clearly, $X < U, \quad \Phi(X) = \langle x^p \rangle$, whence, $\langle x^p \rangle < U$. For $\Delta = 2, \quad \langle x^p \rangle = \langle c \rangle < G$. Let $\Delta > 2$. We set $U_2 = \langle x^p \rangle \times \langle c \rangle$. Since $G$ and, consequently, $U_2$, do not contain dihedral subgroups of order 8, by virtue of [7] $w(U_2) = \langle z \rangle \times \langle c \rangle$. For $U_2 < G, \quad w(U_2) < G$. For $U_3 < G$, according to Theorem 2.2 from [3] $w(U_3) < G$. In $G$, there exists the subgroup $\langle z \rangle w(U_3) = \langle x \rangle \times \langle z \rangle$ which is maximal and, consequently, normal in $U = (\langle x \rangle \times \langle z \rangle) < G$. Therefore, $\Phi(U) = \langle x^p \rangle$ and, consequently, $\langle x^p \rangle < G$. If $g$ is an element from $G$ and $\langle g \rangle < G$, then $\Phi(\langle g \rangle) < G$. Let $g < G$. We set $U_4 = \langle g \rangle M$. If $|g| = p, \quad \Phi(\langle g \rangle) < G$. Let $|g| > p$. Then $U_4$ contains $Y = \langle g \rangle \times \langle f \rangle$, where $f \in M, \quad |f| = p, \quad \Phi(Y) = \langle g^p \rangle$. For $Y < G, \quad \Phi(Y) = \langle g^p \rangle < G$. For $Y < G$, we can assume that $Y = X$ and, as above, we get $\langle g^p \rangle < G$. Thus, we always have $\Phi(\langle g \rangle) < G$.

Let $H$ be a non-Abelian subgroup from $G$. If $H < G$, then $H' < G$. If $H < G$, then we can assume that $H = X$ and, consequently, $H' < G$.

Let $H = \langle u, v \rangle = \langle u \rangle \cup \langle v \rangle, \quad |u| > p, \quad |v| = p$. Since $G$ does not contain dihedral subgroups of order 8, by virtue of [7] $w(H) = w(\langle u \rangle) \times \langle v \rangle$. For $H < G, \quad w(H) < G$. For $H < G, \quad w(H)$ is normal in $G$ in view of Theorem 2.2 from [3]. Thus, assertion (ii) of the lemma is completely proved.

As is known [8], in a finite 2-group $G \Phi(G) = \langle g^2 \rangle$ for all $g$ from $G$ and $G' \leq \Phi(G)$. By the preceding, $\langle g^2 \rangle < G$. It is clear that $C_G(\langle g^2 \rangle) < G$ and $G/C_G(\langle g^2 \rangle)$ is a subgroup from the Abelian group $\text{Aut}(\langle g^2 \rangle)$. Consequently, $C_G(\langle g^2 \rangle)$ contains $G'$ and, therefore, $G' \leq Z(\Phi(G))$. Assertion (iii) of the lemma is proved.

It is easy to show that $G$ does not contain subgroups of quaternions of order 8 and, hence, assertion (i) of the lemma is true. The lemma is proved.

**Theorem 1.** All CDN[\]-groups $G$ of order $p^n$ with commutant $G'$ of the type $(p, p)$ which contains all elements of order $p$ from $G$, where $p$ is a prime number, have the form $G = UX$, $U = \langle a, b \rangle < G, \quad |a| = p^\alpha, \quad |b| = p^\beta, \quad \alpha \in \{2, 3\}, \quad \beta \in \{1, 2\}, \quad |X| > p$ and are exhausted by groups of the following types:

(i) $G = U \times X$, $X$ is the group of quaternions of order 8, $U$ is a cyclic 2-group or the group of quaternions of order 8, $[U, X] \leq w(U)$; for $U' = 1, \quad [U, X] = w(U)$;

(ii) $U = \langle a \rangle \times \langle b \rangle, X = \langle x \rangle, \quad |a| = |b| = |x| = 4, \quad [a, x] = a^2, \quad [b, x] = x^2 = a^2 b^2$;

(iii) $G = \langle \langle a \rangle \times \langle b \rangle \rangle \langle \langle x \rangle \times \langle y \rangle \rangle$, $U = \langle a \rangle \times \langle b \rangle, X = \langle x \rangle \times \langle y \rangle, \quad |a| = |b| = 4, \quad a^2 b^2 = [a, x] = [b, y], \quad b^2 = y^2 = [b, x], \quad a^2 = x^2 = [a, y]$;