ON SOME BASIC PRINCIPLES IN DYNAMIC THEORY OF ELASTIC MATERIALS WITH VOIDS

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ABSTRACT: According to the basic idea of dual-complementarity, in a simple and unified way proposed by the author[1], some basic principles in dynamic theory of elastic materials with voids can be established systematically. In this paper, an important integral relation in terms of convolutions is given, which can be considered as the generalized principle of virtual work in mechanics. Based on this relation, it is possible not only to obtain the principle of virtual work and the reciprocal theorem in dynamic theory of elastic materials with voids, but also to derive systematically the complementary functionals for the eight-field, six-field, four-field and two-field simplified Gurtin-type variational principles. Furthermore, with this approach, the intrinsic relationship among various principles can be explained clearly.

KEY WORDS: dynamic theory of elastic materials with voids, principle of virtual work, reciprocal theorem, variational principle, complementary relation

I. INTRODUCTION

The dynamic theory of elastic materials with voids was established by Cowin and Nunziato[2]. The theory is intended for applications to natural and artificial materials with distributed voids, and some bioengineering materials. Therefore, it plays an important role in the development and application of modern new materials. In recent years, this theory and its application were developed further by Cowin and Puri[3], Passman[4], Cowin[5], Iesan[4], Chandrasekhar[7], Ciarletta and Scalia[8] et al. But some basic principles in dynamic theory of elastic materials with voids, which include the principle of virtual work, the reciprocal theorem and various variational principles, are not yet established systematically.

According to the basic idea of dual-complementarity, in a simple and unified way proposed by the author[1], some basic principles in dynamic theory of elastic materials with voids can be established systematically. In this paper, an important integral relation in terms of convolutions is given, which can be considered as the generalized principle of virtual work in dynamics of small deformation body with voids. Based on this relation, it is possible not only to obtain the principle of virtual work and the reciprocal theorem in dynamic theory of elastic materials with voids, but also to derive systematically the complementary functionals for the eight-field, six-field, four-field and two-field simplified Gurtin-type variational principles by the generalized Legendre transformations given in this paper. Furthermore, with this approach, the intrinsic relationship among various principles can be explained clearly.

II. BASIC EQUATIONS AND CONDITIONS

Consider an elastic body with voids \( V \), and let \( \partial V \) be the boundary of \( V \). The closure of \( V \) is denoted by \( \overline{V} = V \cup \partial V \), and \( \overline{V} \) is a regular region of three-dimensional space. The basic equations, boundary and initial conditions for the dynamic theory of linear elastic materials with voids are:

1. Velocity-displacement relations
(2) Momentum-velocity relations

\[ p_i = \rho v_i \]  \hspace{1cm} (2.2)

The corresponding kinetic energy density and complementary kinetic energy density are as follows:

\[ K (v_i) = \frac{1}{2} \rho v_i v_i \quad K^* (p_i) = \frac{1}{2\rho} p_i p_i \]  \hspace{1cm} (2.2a, b)

(3) Equations of motion\[^{[21]}\]

\[ \sigma_{ij,j} + \rho f_i = \dot{p}_i \quad \text{or} \quad \sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i \]  \hspace{1cm} (2.3a, b)

\[ h_i,i + g + \rho l = \rho k \dot{\phi} \]  \hspace{1cm} (2.4)

where \( \sigma_{ij} \) is the stress tensor, \( \rho \) the density in the reference configuration, \( f_i \) the body force vector, \( h_i \) the equilibrated stress vector, \( g \) the intrinsic equilibrated body force, \( l \) the extrinsic equilibrated body force and \( k \) the equilibrated inertia.

(4) Geometrical equations\[^{[21]}\]

\[ \varepsilon_{ij} = \frac{1}{2} (u_{ij,j} + u_{ji,j}) \]  \hspace{1cm} (2.5)

\[ e_i = \varphi_{,i} \]  \hspace{1cm} (2.6)

where \( \varepsilon_{ij} \) is the strain tensor, \( u_i \) is the displacement vector, \( \varphi \) is the change in volume fraction from the reference volume fraction.

(5) Constitutive equations

\[ \sigma_{ij} = E_{ijkl} \varepsilon_{kl} + \beta_{ijkl} \varepsilon_{kl} + \gamma_{ijkl} \varepsilon_{kl} \varphi \quad \text{or} \quad \varepsilon_{ij} = C_{ijkl} \sigma_{kl} + \alpha_{ijkl} \varepsilon_{kl} + \eta_{ijkl} \varphi \]  \hspace{1cm} (2.7a, b)

\[ h_i = A_{ij} e_j + B_{ij} \varepsilon_{ij} + h_i \varphi \quad \text{or} \quad e_i = B_{ij} h_j + C_{ijkl} \sigma_{kl} + a_i \varphi \]  \hspace{1cm} (2.8a, b)

\[ g = -\omega \dot{\phi} \varphi - \xi \varphi - \gamma_{ij} \varepsilon_{ij} h_i + b_i e_i \quad \text{or} \quad g = -\omega \dot{\phi} + \zeta \varphi + \eta_{ij} \sigma_{ij} h_i + a_i h_i \]  \hspace{1cm} (2.9a, b)

where \( E_{ijkl}, C_{ijkl}, \beta_{ijkl}, \gamma_{ijkl} \), \( A_{ij}, B_{ij}, \gamma_{ij}, \eta_{ij}, a_i, b_i, \omega, \xi, \zeta \) are characteristic coefficients of the material.

The strain energy density and the complementary strain energy density are expressed respectively as

\[ U (\varepsilon_{ij}, e_i, \varphi) = \frac{1}{2} E_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} A_{ij} e_j e_j + \frac{1}{2} \xi \varphi^2 + \beta_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \gamma_{ijkl} \varepsilon_{kl} \varphi + b_i e_i \varphi \]  \hspace{1cm} (2.10a)

\[ U^* (\sigma_{ij}, h_i, \varphi) = \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} + \frac{1}{2} B_{ij} h_j h_j + \frac{1}{2} \zeta \varphi^2 + \alpha_{ijkl} \sigma_{ij} h_i + \eta_{ij} \sigma_{ij} \varphi + a_i h_i \varphi \]  \hspace{1cm} (2.10b)

(6) Boundary conditions\[^{[2,6]}\]

(i) Displacement boundary condition

\[ u_i = \bar{u}_i \quad \text{on} \quad \partial V_U \times [0, \infty) \]  \hspace{1cm} (2.11)

(ii) Traction boundary condition

\[ T_i = \sigma_{ij} n_j = \bar{T}_i \quad \text{on} \quad \partial V_T \times [0, \infty) \]  \hspace{1cm} (2.12)

(iii) Boundary condition for the change in volume fraction

\[ \varphi = \bar{\varphi} \quad \text{on} \quad \partial V_\varphi \times [0, \infty) \]  \hspace{1cm} (2.13)

(iv) Equilibrated stress vector boundary condition

\[ h = h_i n_i = \bar{h} \quad \text{on} \quad \partial V_h \times [0, \infty) \]  \hspace{1cm} (2.14)

where \( \bar{u}_i, \bar{T}_i, \bar{\varphi}, \bar{h} \) are prescribed functions, and

\[ \partial V = \partial V_U \cup \partial V_T = \partial V_\varphi \cup \partial V_h \quad \partial V_U \cap \partial V_T = \partial V_\varphi \cap \partial V_h = 0 \]

(7) Initial conditions

\[ \bar{u}_0 (x) = u_i (x, 0) = \bar{u}_i (x) \quad \rho_0 (x) = p_i (x, 0) = \bar{p}_0 (x) \]  \hspace{1cm} (2.15a, b)

\[ \varphi_0 (x) = \varphi (x, 0) = \bar{\varphi}_0 (x) \quad \dot{\varphi}_0 (x) = \dot{\varphi} (x, 0) = \bar{\dot{\varphi}}_0 (x) \quad x \in \overline{V} \]  \hspace{1cm} (2.15c, d)