SOME INEQUALITIES FOR GRADIENTS OF HARMONIC FUNCTIONS

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For a function $u(x, y)$ harmonic in the upper half-plane $y > 0$ and represented by the Poisson integral
of a function $v(t) \in L_2(-\infty, \infty)$, we prove that the inequality

$$|\text{grad } u(x, y)|^2 \leq \frac{1}{4\pi y^2} \int_{-\infty}^{\infty} v^2(t) \, dt$$

is true. A similar inequality is obtained for a function harmonic in a disk.

Theorem 1. Suppose that a function $u(x, y)$ harmonic in the upper half-plane $-\infty < x < \infty, \, y > 0$ can be represented by the Poisson integral

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{v(t) \, dt}{(x-t)^2 + y^2},$$

where $v(t) \in L_2(-\infty, \infty)$. Then the inequality

$$|\text{grad } u(x, y)|^2 \leq \frac{1}{4\pi y^2} \int_{-\infty}^{\infty} v^2(t) \, dt$$

is true at any point $(x, y)$ of the half plane.

Theorem 2. Suppose that a function $u(z), \ z = x + iy, \ harmonic \ in \ a \ disk \ |z| < 1$ can be represented by the Poisson integral

$$u(z) = \Re \frac{1}{2\pi} \int_{0}^{2\pi} v(\theta) \frac{z + \zeta}{\zeta - z} \, d\theta, \quad \zeta = e^{i\theta},$$

where $v(\theta) \in L_2(0, 2\pi)$. Then the inequality

$$|\text{grad } u(z)|^2 \leq \frac{2}{\pi(1 - |z|^2)^2} \int_{0}^{2\pi} v^2(\theta) \, d\theta$$

is true at any internal point $z$ of the disk.

Proof of Theorem 1. Let us find partial derivatives of function (1) at a fixed point $(x, y)$ of the upper half-plane and represent them in the form of a scalar product of the corresponding elements of the real Hilbert space $H = L_2(-\infty, \infty)$ as


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\[
\frac{\partial u}{\partial x} = (a, v), \quad \frac{\partial u}{\partial y} = (b, v),
\]

where

\[
a(t) = \frac{2(t - x)y}{\pi((x-t)^2 + y^2)^2} \in L_2(-\infty, \infty),
\]

\[
b(t) = \frac{(x - t)^2 - y^2}{\pi((x-t)^2 + y^2)^2} \in L_2(-\infty, \infty).
\]

In this case, \( \operatorname{grad} u(x, y) = ((a, v); (b, v)) \). Note several properties of elements \( a \) and \( b \) from \( H \) [1]:

(i) \( (a, b) = 0 \);
(ii) \( ||a||^2 = ||b||^2 = \frac{1}{4\pi y^3} \).

Let us write the Bessel inequality [2, p. 177] for an element \( v \in H \) and orthogonal elements \( a \) and \( b \):

\[
||v||^2 \geq \frac{1}{||a||^2} ||a||^2 + \frac{1}{||b||^2} ||b||^2 = \frac{1}{||a||^2} ||\operatorname{grad} u(x, y)||^2.
\]

Hence,

\[
||\operatorname{grad} u(x, y)||^2 \leq ||a||^2 ||v||^2 = \frac{1}{4\pi y^3} \int_{-\infty}^{\infty} v^2(t) \, dt.
\]

Theorem 1 is proved.

**Proof of Theorem 2.** Let us find partial derivatives of function (2) at a point \( z = x + iy \) and represent them in the form of a scalar product of the corresponding elements of the complex Hilbert space \( H = L_2(0, 2\pi) \) as

\[
\frac{\partial u}{\partial x} = \operatorname{Re}(a, v), \quad \frac{\partial u}{\partial y} = \operatorname{Re}i(a, v) = -\operatorname{Im}(a, v),
\]

where

\[
a(\theta) = \frac{\zeta}{\pi(\zeta - z)^2} \in L_2(0, 2\pi), \quad \zeta = e^{i\theta}.
\]

Therefore,

\[
(a, v) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}, \quad |(a, v)| = ||\operatorname{grad} u(z)||
\]

and we can use the Cauchy–Buniakowski inequality \( |(a, v)| \leq ||a|| \cdot ||v|| \). In this case, we have