GENERAL SOLUTION FOR INTERACTION OF SOLITARY WAVES INCLUDING HEAD-ON COLLISIONS

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ABSTRACT: A corrected version of the Boussinesq equation for long water waves is derived and its general solution for interaction of any number of solitary waves, including head-on collisions, is given. For two solitary waves in head-on collision (which includes the case of normal reflection) the results agree with the experiments known.

KEY WORDS: solitary waves, Boussinesq equation, head-on collision

I. INTRODUCTION

Ever since Boussinesq (1872)\(^1\) derived his single equation model for long water waves in a rectangular channel, it has been known as the Boussinesq equation, in addition to the better known set of two equations also bearing his name, and various authors (Hirota (1973)\(^2\), Oikawa and Yajima 1973\(^3\), Yih (1993)\(^4\), among others) have given solutions of this single equation, which are in general agreement with each other. However, Byatt-Smith (1971)\(^5\) has given an equation different from Boussinesq’s, and obtained a result for solitary-wave reflection which gives an implicit phase shift opposite to that predicted by these authors and in agreement with the experiments of Chan and Street (1970)\(^6\) and of Maxworthy (1976)\(^7\), as well as the analysis of Power and Chwang (1984)\(^8\). Recently Wu (1994)\(^9\) gave an equation, in terms of the velocity potential, for long waves in channels of variable width and depth, which for rectangular channels reduces to a corrected version of the Boussinesq equation. Wu’s equation shows that the nonlinear terms contain time derivatives of the dependent variable, as indeed does the more complicated equation of Byatt-Smith\(^5\).

As far as we know, no one has given the general solution for the corrected Boussinesq equation for the interaction of any number of solitary waves, including head-on collisions. The solution is provided by the method of Yih (1993, which will henceforth be referred to as Y)\(^4\), and will be given here. In order to use Yih’s method of solution, a corrected Boussinesq equation in terms of \(\eta\) will be derived here. It will be shown that the nonlinear terms for the interaction equation (in terms of \(\tilde{\eta}\), the displacement induced by the presence of left-going waves and right-going waves) are just those given in Y multiplied by \(-1/3\). The development is therefore much shortened by using the corrected Boussinesq equation in
terms of $\eta$. The results are then easily obtained, and are in agreement with the experiments of Maxworthy's [7] and of Chan and Street's [6] as regards phase shifts and amplitudes.

II. ANALYSIS

We start with the equations

$$h_t + (uh)_x = 0 \quad (1)$$

$$u_t + uu_x + gh_x + \frac{c_0^2 h_o}{3} h_{xxx} = 0 \quad (2)$$

in which $u$ is the horizontal velocity and $h$ is the water depth, both assumed to be functions only of the time $t$ and the horizontal distance $x$, $g$ is the gravitational acceleration, $h_o$ is the depth of water when undisturbed by waves, and

$$c_o = (gh_o)^{1/2} \quad (3)$$

is the linear long-wave velocity. These equations are given on p.461 of Whitham's book (1974) [1]. The first is the equation of continuity, and the second is the equation of motion, with dispersion taken into account to the desired order of approximation. The last term in (2) is sometimes written as $(h_o/3)h_{xxt}$. Since it is linear and involves the substitution of the second derivative of $h$ with respect to $x$ by that of $t$ only, the replacement is immaterial even when right-going waves and left-going ones coexist.

We shall assume

$$h = h_o + \eta \quad (4)$$

where $\eta$ is the surface displacement, assumed to be small in comparison with $h_o$. We shall assume $h_o$ to be constant, and shall use it later as the lengthscale. Then (1) and (2) become

$$\eta_t + [u(h_o + \eta)]_x = 0 \quad (5)$$

$$u_t + uu_x + g\eta_x + \frac{c_o^2 h_o}{3} \eta_{xxx} = 0 \quad (6)$$

Differentiating (5) with respect to $t$, we have

$$\eta_{tt} + h_o u_{xt} + (u\eta)_{xt} = 0$$

Substituting (6) into this, we obtain

$$\eta_{tt} - h_o (uu_x + g\eta_x + \frac{c_o^2 h_o}{3} \eta_{xxx})_x + (u\eta)_{xt} = 0$$

or

$$\eta_{tt} - c_0^2 \eta_{xx} + N(x, t) - \frac{c_o^2 h_o^2}{3} \eta_{xxxx} = 0 \quad (7)$$

where the nonlinear terms $N(x, t)$ are given by

$$N(x, t) = -h_o (uu_x)_x + (u\eta)_{xt} = -h_o u_x u_x - h_o uu_{xx} + u_x \eta_x + u \eta_{xt} + u_t \eta_x \quad (8)$$

The dominant terms in the linear parts of (5) and (6) give

$$\eta_t + h_o u_x = 0 \quad (9)$$

$$u_t + g\eta_x = 0 \quad (10)$$