ON PROPERTIES OF HYPERCHAOS: CASE STUDY*

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ABSTRACT: Some properties of hyperchaos are exploited by studying both uncoupled and coupled CML. In addition to usual properties of chaotic strange attractors, there are other interesting properties, such as: the number of unstable periodic points embedded in the strange attractor increases dramatically increasing and a large number of low-dimensional chaotic invariant sets are contained in the strange attractor. These properties may be useful for regarding the edge of chaos as the origin of complexity of dynamical systems.

KEY WORDS: hyperchaos, strange attractor, unstable periodic point, pattern formation

1 INTRODUCTION

The dynamical system with many degrees of freedom often exhibit complicated behaviors. Recently, this subject has drawn researchers more attention. The dynamical behavior of so called "hyperchaos" with more than one positive Lyapunov exponents often appears in the numerical simulation of a multi-degree-of-freedom system. So does in the numerical investigation on coupled maps of lattice (CML)\(^1\). In the analysis of zigzag phenomenon in CML, we have derived a two-dimensional map in which the states with two positive Lyapunov exponents occur. We conjecture that the hyperchaotic states of a multi-degree-of-freedom system has exerted great effects on the origin of complicated behavior. Therefore, it is necessary to study hyperchaos in multi-degree-of-freedom systems.

In order to simplify the discussion and to avoid mathematical difficulties, we introduce a simple uncoupled two-dimensional logistic map at first

\[
L : \begin{cases}
    x_{n+1} = 4x_n(1 - x_n) \\
    y_{n+1} = 4y_n(1 - y_n)
\end{cases}
\]

where \(L : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]\). Model (1) may contain certain primary information of hyperchaos. Thus some important properties of hyperchaos can be summarised by analysing

Received 13 July 1998, revised 7 July 1999

* The project supported by the National Natural Science Foundation of China
Eq.(1)). Then a coupled two-dimensional map in [2] is treated as an example to illustrate these properties of hyperchaos by both theoretic analysis and numerical simulations.

2 THE DYNAMICAL BEHAVIOR AND THE STRUCTURE OF STRANGE ATTRACTOR OF MAP L

The dynamical behaviors of map \( L \) are given as follow.

**Property 1:** \( L \) is chaotic in the sense of Marotto\(^{[3-5]} \).

**Proof:** Logistic map \( f : x_{n+1} = 4x_n(1 - x_n) \) is chaotic in the sense of Marotto, namely in a small neighborhood of the fixed point \( x_0 \), there is a point \( \tilde{x} \) and a positive \( M > 0 \), such that \( f^M(\tilde{x}) = \tilde{x}_0 \) and \( |Df^M(\tilde{x})| \neq 0 \). Now, \((x_0, y_0)\) is a fixed point of map \( L \), where \( x_0 = y_0 = \tilde{x}_0 \). It is obvious that a point \((x, y)\), where \( x = y = \tilde{x} \), can be found in a neighborhood of the fixed point, such that \( L^M(x, y) = (x_0, y_0) \) and \( |DL^M(x, y)| \neq 0 \). So \( L \) is chaotic in the sense of Marotto.

For Eq.(1), \( \forall x \in [0, 1] \) and \( \forall y \in [0, 1] \), a corresponding symbol series in \( \Sigma_2 \) can be defined. Then the shift map \( \sigma : \Sigma_2 \rightarrow \Sigma_2 \times \Sigma_2 \) can be set, namely \( \forall s = (.s_0s_1s_2\cdots) \in \Sigma_2 \) and, \( \forall t = (.t_0t_1t_2\cdots) \in \Sigma_2 \)

\[ \sigma(s, t) = (\sigma s, \sigma t) = (s', t') \]

where \( s' = (.s_1s_2s_3\cdots) \) and \( t' = (.t_1t_2t_3\cdots) \).

**Property 2:** \( L_{[0, 1] \times [0, 1]} \sim \sigma |\Sigma_2 \times \Sigma_2 | \).

**Proof:** Property 2 is easily derived from the fact that map \( f : x_{n+1} = 4x_n(1 - x_n) \) possesses property \( f |_{[0, 1]} \sim |\Sigma_2 \times \Sigma_2 | \).

As a result of the dynamical behavior of \( \sigma \) on \( \Sigma_2 \times \Sigma_2 \), \( L \) is topologically transitive and sensitively dependent on initial conditions on \([0, 1] \times [0, 1] \).

**Property 3:** \( L \) has two positive Lyapunov exponents.

**Proof:** The result is obvious.

According to properties 1-3, the dynamical behavior of \( L \) is hyperchaotic. Therefore it has a strange attractor with hyperchaotic behavior on \([0, 1] \times [0, 1] \). Based on the characters of one-dimensional logistic map, this strange attractor must spread all over the region \([0, 1] \times [0, 1] \). This strange attractor has the following properties.

**Property 4:** \( A = \overline{W^u}(p) \), where \( p \) is an expanding fixed point of \( L \) on \([0, 1] \times [0, 1] \).

**Proof:** \( A \) is an invariant set of \( L, p \in A \), so \( \overline{W_{loc}^u}(p) \subset A \)

\[ L^n(\overline{W_{loc}^u}(p)) \subset A \]

\[ \bigcup_{n \geq 0} L^n(\overline{W_{loc}^u}(p)) \subset A \]

Hence \( \overline{W^u(p)} \subset A \). On the other hand, \( L \) is topological transitive on \( A \), and \( W^u(p) \) is an open set, so \( A \subset \bigcup_{n \geq 0} L^n(\overline{W_{loc}^u}(p)), \) namely \( A \subset \overline{W^u(p)} \). Thus is \( A = \overline{W^u(p)} \) proved.

**Property 5:** The periodic points in \( A \) is dense.

**Proof:** \( \forall (x_0, y_0) \in [0, 1] \times [0, 1], V \) is an \( \varepsilon \)-neighborhood of \((x_0, y_0) \). \( U \) is also a neighborhood of \((x_0, y_0) \), of which the diameter is less than \( \varepsilon / \sqrt{2} \). Based on the fact that the periodic points of logistic map with \( \lambda = 4 \) is dense in \([0, 1] \), there exist periodic points \( \tilde{x} \) and \( \tilde{y} \) of logistic maps \( x_{n+1} = 4x_n(1 - x_n) \) and \( y_{n+1} = 4y_n(1 - y_n) \) in the \( U \cap \{x = x_0\} \) and \( U \cap \{y = y_0\} \). Then \((\tilde{x}, \tilde{y}) \in V \) and \((\tilde{x}, \tilde{y}) \) is a periodic point of \( L \). Since \( \varepsilon \) is arbitrary small, the periodic points of \( L \) is dense in \( A \).

From the above discussion, the geometric structure of the hyperchaotic strange attractors is almost same as that of usual chaotic strange attractors with only one positive Lyapunov exponent\(^{[6]} \). The difference between them is shown in their phase portraits. The