ON THE CLASSIC NONHOLONOMIC DYNAMICS

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ABSTRACT: For first-order nonlinear nonholonomic systems, the present paper proves that the $d\delta$ operations are commutative and derives the equation of motion without making use of the additional Appell-Chetaev condition. This equation of motion coincides with the equation of "Vacco dynamics".

KEY WORDS: nonholonomic dynamics, commutativity of $d\delta$ operations, Appell-Chetaev condition, Vacco dynamics.

Since Hertz clearly introduced the concept of nonholonomic constraints in 1894, the classic nonholonomic dynamics has gradually become a branch of analytic mechanics. More and more scientists involved in its investigation and established different kinds of equations of motion. But the following two basic problems remain unsolved and are still in dispute:

1) Are the operations — differentiation $d$ and variation $\delta$ — commutative?

2) Does the Appell-Chetaev condition for virtual displacements, which is automatically satisfied for linear nonholonomic constraints, apply also for nonlinear cases?

About these two problems, opinions are widely divided. Mei Fengxiang in [1, 2] gave the following brief summary: "Historically, there have been two views on the form of the interchange relation. According to the first view, $d\delta$ operations are always commutative both for holonomic and for nonholonomic systems; according to the second one, however, the $d\delta$ commutativity holds only for holonomic systems. These two views dispute strongly, but the last one gains in support"; "Although no question arose about the applicability of Appell-Chetaev definition, everyone declares cautiously that he is dealing with the nonlinear nonholonomic constraints of Chetaev-type". In fact, some authors studied also "nonholonomic systems of non-Chetaev-type", substituting the partial derivatives of the constraint functions in the Chetaev condition by some other functions, i.e. using new condition instead of Chetaev condition. This approach is still in the framework of imposing additional conditions on virtual displacements. These authors didn't care: What should these functions in general be? What should their connection to the constraints of the system be?

The undefiniteness of these two basic problems cast a shadow over this branch of mechanics and put it in an unsatisfactory situation. The present paper tries to clarify these problems for first-order nonlinear nonholonomic systems with the following statements:

Received 25 November 1988.

1) Project supported by the Science-Technology Foundation for Universities.

2) At present, Dept. of Mathematics, Xiantan Normal School.
(1) $d \cdot \delta$ operations are commutative;
(2) It is not necessary to impose additional conditions on virtual displacements.

First we prove strictly

**Theorem** $d \cdot \delta$ operations are commutative for first-order nonlinear nonholonomic systems. □

Hence, the commutativity is not a question of view-point, but a necessary conclusion of logical reasoning. For the proof, two lemmata are needed.

Let $q_1, \ldots, q_n$ be generalized coordinates of the system. It has $m$ independent first-order nonlinear nonholonomic constraints:

$$f_i(q_1, \ldots, q_n, \dot{q_1}, \ldots, \dot{q_n}, t) = 0 \quad i = 1, \ldots, m \quad (1)$$

Assume that every $f_i$ possesses the later needed continuous partial derivatives.

**Lemma I** (see [3]) Let the continuously differentiable function $p(t)$ on interval $[a, b]$ be a variationally independent variable. Then

$$\frac{d}{dt} \delta p(t) = \delta \dot{p}(t) \quad (2)$$

**Proof** Obviously, it suffices to show that (2) holds in the neighborhood of any $t \in [a, b]$. As shown in the Fig. 1, $y = p(t)$ and $y = p(t) + \delta p(t)$ are two curves, sufficiently close, i.e., possessing at least a first-order proximity degree. Assume that the ordinate of point $A$ is $p(t)$, then the ordinate of point $A'$ is $p(t) + \delta p(t)$. By means of Taylor formula, the ordinate of point $B$ may be expressed as

$$B: \quad p(t + \Delta t) = p(t) + \dot{p}(t)\Delta t + O(|\Delta t|^2)$$

The ordinate of point $B'$ may be calculated in two ways. The first way is to start from point $B$, then

$$B': \quad p(t + \Delta t) + \delta p(t + \Delta t)$$
$$= p(t) + \dot{p}(t)\Delta t + O(|\Delta t|^2) + \delta(p(t) + \dot{p}(t)\Delta t + O(|\Delta t|^2))$$
$$= p(t) + \dot{p}(t)\Delta t + \delta p(t) + \delta\dot{p}(t)\Delta t + O(|\Delta t|^2) \quad (3)$$

Here $\Delta t$ is taken as fixed. The another way is to start from point $A'$. Then, using Taylor formula, we have

$$B': \quad p(t) + \delta p(t) + \frac{d}{dt} (p(t) + \delta p(t))\Delta t + O(|\Delta t|^2)$$
$$= p(t) + \delta p(t) + \dot{p}(t)\Delta t + \frac{d}{dt} (\delta p(t))\Delta t + O(|\Delta t|^2) \quad (4)$$