NONLINEAR TRAVELING WAVES IN A COMPRESSIBLE
MOONEY-RIVLIN ROD
I. LONG FINITE-AMPLITUDE WAVES

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ABSTRACT: In literature, nonlinear traveling waves in elastic circular rods have only been studied based on single partial differential equation (pde) models, and here we consider such a problem by using a more accurate coupled-pde model. We derive the Hamiltonian from the model equations for the long finite-amplitude wave approximation, analyze how the number of singular points of the system changes with the parameters, and study the features of these singular points qualitatively. Various physically acceptable nonlinear traveling waves are also discussed, and corresponding examples are given. In particular, we find that certain waves, which cannot be counted by the single-equation model, can arise.

KEY WORDS: hyperelastic rod, nonlinear traveling waves, solitary waves, periodic waves

1 INTRODUCTION

It is well-known that waves propagating along a rod are subject to dispersion due to the transverse dimension of the rod\cite{1-3}. When the amplitude of the wave is reasonably large, nonlinearity may become as important as dispersion. Therefore both dispersion and nonlinearity should be considered for these waves.

Nonlinear dispersive waves have been investigated through the rod equations by taking into account nonlinearity. Wright studied these nonlinear waves and pointed out that a variety of traveling waves can arise in a hyperelastic rod by assuming that the strain-energy function is a function of the axial displacement, the radial strain and its derivative\cite{4}. Coleman and Newman derived the one-dimensional rod equation from the three-dimensional rod theory and obtained explicit results for a rod composed of a neo-Hookean material\cite{5}. Their paper dealt with smooth traveling waves only, while for a Mooney-Rivlin elastic rod, some singular waves may arise. A comprehensive study on these types of singular waves was carried out in Ref.[6]. In particular, it was shown that solitary shock waves can appear. In Ref.[7], it was shown for the first time that kink waves can propagate along an incompressible hyperelastic rod and their mathematical descriptions were given.

Nonlinear dispersive waves have also been investigated through the rod equations by assuming that the wave amplitude is small-but-finite and the wave length is long (i.e., the waves concerned are long finite-amplitude waves). Nariboli has shown that the KdV equation can be considered as the model equation for uni-directional long finite-amplitude waves in an elastic rod\cite{8}. In order to study bi-directional waves, Soerensen et al. and Clarkson et al. have derived an improved Boussinesq equation\cite{9-11}. However, the approximation used in Refs.[9-11] (i.e., the radial strain is a function of the axial strain) is questionable. Samsonov\cite{12} used a similar assumption to derive a single double dispersive equation as the model equation. Without using such an approximation, Cohen and Dai have obtained another set of model equations for bi-directional waves in problems such as head-on

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collision or reflection of solitary waves\cite{13}. As for an incompressible Mooney-Rivlin material, it was shown in Ref.\cite{14} that solitary shock waves can also arise in a compressible hyperelastic material.

The research reported in Refs.\cite{8~13} analyzed disturbances only for an initially stress-free rod and long finite-amplitude waves. In Ref.\cite{15}, Dai considered disturbances in an initially stretched or compressed rod for both long finite-amplitude waves and finite-length and finite-amplitude waves. He assumed that the rod is composed of a compressible Mooney-Rivlin material. Many polymeric elastomers, including properly treated natural rubber, can be regarded as Mooney-Rivlin materials. By the reductive perturbation technique, he derived a new type of nonlinear dispersive equations which includes extra nonlinear terms involving second-order and third-order derivatives. When the rod is composed of a compressible neo-Hookean material, this equation is reduced to the so-called Benjamin-Bona-Mahony equation\cite{16}.

To the authors' knowledge, nonlinear traveling waves in elastic circular rods have only been studied based on single-equation models, although for compressible materials physically the axial deformation and radial deformation should be regarded as two independent unknown. In this paper, we shall study nonlinear traveling waves in a compressible Mooney-Rivlin rod by using the two-equation system derived in Ref.\cite{15}. In the first part of this series of two papers, we shall only consider long finite-amplitude waves, and in a sequel paper, finite-length and finite-amplitude waves will be studied. The most important findings of this work are: while for a single-equation model there is only one type of phase plane (consequently there exist only one type of solitary waves and one type of periodic waves), here we find that for the two-equation model there are three types of phase planes (consequently there exist two types of solitary waves and three types of periodic waves). Thus, to use a single-equation model, certain nonlinear traveling waves will be missed out. This suggests that for nonlinear dispersive waves in compressible elastic rods one should use a two-equation model. Here, through a bifurcation analysis, we also manage to determine the precise parameter domain in which each type of traveling waves can arise. The profiles for all these types of waves are presented.

This paper is arranged as follows. In Section 2, a Hamiltonian system, which describes various nonlinear traveling waves, is derived from the model equations given in Ref.\cite{15}. The number of singular points and their qualitative behavior are analyzed in detail in Sections 3 and 4. In Section 5, all practically interesting cases are discussed based on an asymptotic analysis. Finally, various physically acceptable nonlinear traveling waves are discussed, and corresponding examples are given.

2 THE BASIC EQUATIONS

For an elastic circular rod composed of a compressible Mooney-Rivlin material, the dimensionless model equations for long finite-amplitude waves are\cite{16}

\begin{equation}
\frac{\rho c^2}{\mu} w_{tt} + \eta_1 w_{xx} + \eta_2 u_x + \epsilon \left( -\eta_4 w_x w_{xx} + \frac{\eta_2}{\alpha} u w_{xx} + \eta_5 u w_x + \frac{\eta_2}{\alpha} u_x w_x \right) = 0 \tag{2.1}
\end{equation}

\begin{equation}
\eta_3 u + \eta_2 w_x + \epsilon \left( \eta_0 u^2 + \eta_5 u w_x + \frac{\eta_2}{2\alpha} w_x^2 \right) + \nu \left( \frac{\rho c^2}{2\mu} u_{xx} - \frac{1}{4} \eta_0 u w_x \right) = 0 \tag{2.2}
\end{equation}

where \( w \) and \( u \) are, respectively, the axial and radial disturbances superimposed in a uniformly stretched state with the radial stretch equal to \( \lambda \) and the constant coefficients are given by

\[
\begin{align*}
\eta_0 &= (1 - 2\beta)\lambda^2 + 1 + 2\beta \\
\eta_1 &= \frac{1}{2} \left[ 1 + 2\beta + \frac{2k + 3 - 2\beta}{\alpha^2} + 2(1 - 2\beta)\lambda^2 + 2k\lambda^4 \right] \\
\eta_2 &= 2(1 - 2\beta)\lambda \alpha + 4k\lambda^3 \alpha \\
\eta_3 &= 1 + 2\beta + 3(1 - 2\beta)\lambda^2 + 6k\lambda^2 \alpha^2 + \\
&\quad (1 - 2\beta)\alpha^2 + \frac{2k + 3 - 2\beta}{\lambda^2} \\
\eta_4 &= \frac{2k + 3 - 2\beta}{\alpha^2} \\
\eta_5 &= 12k\lambda^2 \alpha + 2(1 - 2\beta)\alpha \\
\eta_6 &= 3(1 - 2\beta)\lambda + 6k\lambda^2 \alpha^2 - \frac{2k + 3 - 2\beta}{\lambda^3} \\
\eta_7 &= 1 - 2\beta + 2k\alpha^2 + \frac{2k + 3 - 2\beta}{\lambda^4} \\
\nu &= \frac{h}{l} \\
\epsilon &= \frac{a^2}{l^2} \\
\frac{\rho c^2}{\mu} &= \frac{(\eta_1 \eta_3 - \eta_2^2)}{\eta_3} \\
\alpha &= \frac{1}{\lambda} \sqrt{\frac{2k + 3 - 2\beta - (1 + 2\beta)\lambda^2 - (1 - 2\beta)\lambda^4}{2k\lambda^2 + (1 - 2\beta)}}
\end{align*}
\]