NUMERICAL METHOD OF POTENTIALS IN THE PLANE PROBLEM OF
QUASISTATIC THERMOELASTICITY

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An algorithm for the numerical solution of the plane problem of uncoupled quasistatic
thermoelasticity is constructed and implemented in this paper on the basis of an indirect
formulation of the method of boundary integral equations. The force vector and the condition
of convective heat transfer from the environment are given on the boundary of a finite or
infinite multiconnected domain. The paper is oriented toward solving geomechanical problems
on determining the stress-strain state in the neighborhood of mining drifts being exploited
under heat transfer conditions from a surrounding mass (large depths, permafrost) when taking
account of temperature stresses is necessary. The algorithms developed can also be utilized
in other numerous applied methods of the theory of thermoelasticity and heat conduction.

1. FORMULATION OF THE PROBLEM

Let D denote a multiconnected domain in the \((x_1, x_2)\) plane bounded by simple closed
contours \(\Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_m = \Gamma\) (Fig. 1). The contour \(\Gamma_0\) encloses all the rest, when it is missing
the domain D is infinite.

In the absence of mass forces and heat sources, the system of differential equations of
uncoupled quasistatic thermoelasticity has the form [1]

\[
\mu \Delta u + (\lambda + \mu) \text{grad} \text{div} u = \gamma \text{grad} \theta, \\
\Delta \theta = \frac{1}{\kappa} \frac{\partial \theta}{\partial t},
\]

where \(u(x, t)\) is the displacement vector with components \(u_1, u_2, x = (x_1, x_2)\) is a point on
the plane, \(\theta(x, t)\) is the temperature measured from the initial state, \(\lambda, \mu\) are Lame1 con-
stants, \(\gamma = \alpha(3\lambda + 2\mu)\), \(\alpha\) is the coefficient of linear thermal expansion, and \(\kappa\) is the coef-
ficient of thermal diffusivity. The stresses and strains are related by the known Duhamel–
Neumann relationships.

The force vector on an area with normal \(n\) at the point \(x\) is expressed in terms of the
displacement and the temperature

\[
p_i(x, t) = T_i \left( \frac{\partial}{\partial x_i}, n(x) \right) u_i(x, t) - \eta n_i(x) \theta(x, t), \quad i = 1, 2,
\]

where \(T_i \left( \frac{\partial}{\partial x_i}, n(x) \right) = \lambda n_i(x) \frac{\partial}{\partial x_j} + \mu n_i(x) \frac{\partial^2}{\partial x_i \partial x_j} + \mu \delta_{ij} \frac{\partial}{\partial x_j} \), \(i, j = 1, 2\), are the stress operator components
[2, 3].

It is required to find the solution of the system (1) and (2) under the following ini-
tial and boundary conditions:

\[
\theta(x, 0) = 0, \quad x \in D \cup \Gamma,
\]

\[
\frac{\partial \theta}{\partial n(y)} + h(y, t) \theta(y, t) = h(y, t) \Phi(y, t), \quad y \in \Gamma, \quad t > 0,
\]

\[
p_i(y, t) = f_i(y, t), \quad y \in \Gamma, \quad t > 0, \quad i = 1, 2,
\]
Fig. 1

where \( \partial \theta / \partial n(y) \) is the derivative with respect to the normal external to the domain (see Fig. 1), \( h(y, t) \) is the heat transfer coefficient, \( \theta(y, t) \) is the temperature of the surrounding medium, and \( f_i(y, t) \) are components of the external force vector. In the case of an infinite domain \( D \) zero conditions at infinity are also given for the temperature and stress \( \theta(\infty, t) = 0, \sigma_{ij}(\infty, t) = 0. \) The boundary contours and the functions given on them are assumed sufficiently smooth.

2. SOLUTION OF THE PROBLEM

The heat-conduction equation (2) with the conditions (4) and (5) is solved to find the temperature field that is determined independently of the displacement field. The temperature found is used in solving (1), for which the time plays a parameter.

We seek the temperature in the form of a simple-layer thermal potential with unknown density \( \varphi_2(\xi, t) \) [4]

\[
\theta(x, t) = \int \frac{d\xi}{\pi} \theta^*(x, \xi, t) \varphi_2(\xi, t) d\xi,
\]

where \( \theta^*(x, \xi, t) = \frac{1}{2\pi} \frac{r}{(t-\xi)^2} \exp \left( -\frac{r^2}{4\pi(t-\xi)} \right) \) is the fundamental solution of (2). The potential (7) satisfies the condition (4). In conformity with the boundary properties of the normal derivative of the potential (7) and the boundary condition (5) to determine the density \( \varphi_2 \) we obtain the integral equation

\[
\varphi_3(y, t) = \frac{1}{4\pi} \int \frac{d\xi}{\pi} \int \frac{\varphi_2(\xi, t)}{(t-\xi)^2} \exp \left( -\frac{r^2}{4\pi(t-\xi)} \right) r \cos (\varpi, n) ds t \int \frac{\varphi_2(\xi, t)}{t-\xi} \exp \left( -\frac{r^2}{4\pi(t-\xi)} \right) ds = h(y, t) \theta(y, t),
\]

where \( n \) is the external normal at the point \( y \in \Gamma \), and the vector \( \vec{r} \) is directed from \( \xi \) to \( y \).

Substituting \( \theta = \theta^0 \) into (1) and solving it in cylindrical coordinates as an ordinary differential equation, we find the fundamental solution of (1) corresponding to the temperature \( \theta^0 \):

\[
\varphi^0_i(x, \xi, t, \varpi) = \frac{\lambda}{(\lambda + 2\mu)} \frac{r_i - r_j}{r^2} \left[ 1 - \exp \left( -\frac{r^2}{4\pi(t-\xi)} \right) \right], i = 1, 2.
\]

The solution (9) can also be obtained by other means [5].

We seek the displacement in the form

\[
u_i(x, t) = U_i(x, t) + V_i(x, t), i = 1, 2,
\]

where \( U_i(x, t) = \frac{1}{\pi} \int \frac{\delta_{ij}(x-\xi)}{\Gamma_i(x-\xi) \varphi_j(\xi, t) d\xi}, j = 1, 2 \), is the simple-layer elastopotential with unknown density \( \varphi = (\varphi_i, \varphi_2) \):

\[
\Gamma_i(x-\xi) = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \delta_{ij} \ln r - \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)} \frac{(x_i - \xi_i)(x_j - \xi_j)}{r^2}
\]

is the matrix of fundamental solutions of isothermal elasticity.