STRESS DISTRIBUTION DURING COMPRESSION OF CYLINDRICAL BODIES

S. K. Ruppeneit and K. V. Ruppeneit

The solution of the problem of cylindrical body compression has numerous technical applications and can be utilized in computing the carrying capacity of pillars and columns, in interpreting the results of laboratory investigations of the mechanical properties of rock and soils. A necessary hypothesis for the solution of the applied problems mentioned is the presence of analytic solutions on the axisymmetric deformation of cylindrical bodies whose material is subject to Hooke's law. As is known, the solution of this class of problems reduces to finding certain functions that are particular solutions of the strain compatibility equation written in the form [1]

$$\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2}\right) \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right) = 0,$$

where $\varphi(r, z)$ is the stress function that we seek as a polynomial of the form

$$\varphi_{m,n}(r, z) = X_1 \left(\sum \frac{a_{m,n}^m z^n}{n + 2}\right),$$

where $m \geq 2$ is a natural integer, $n = 0$ or 1, $a_{ij}$ are internal coefficients of the polynomial, $X_1$ is an external free coefficient to be determined from the equations to satisfy the boundary conditions, and the coefficient $a_{m,n}$ in the highest term can be taken as $a_{m,n} = 1$ without loss of generality in the results.

Certain methods permitting the construction of power-law polynomials, i.e., finding such values of $a_{ij}$ for which $\varphi_{m,n}$ satisfy (1) identically, have been proposed in [2-4]. We propose to construct the polynomial $\varphi_{m,n}$ from formulas whose derivation, although elementary, is sufficiently awkward [5].

Solutions of the harmonic equation are the polynomials

$$\varphi_{m,n}^H = r^m z^n + \frac{m(m-1)z^{n+2}}{} + \frac{m(m-4)z^{n+4}}{} + \frac{m(m-6)z^{n+6}}{} + \ldots,$$

and of the biharmonic, the polynomials

$$\varphi_{m,n}^{II} = r^m z^n + \frac{m(m-2)z^{n+2}}{} + \frac{m(m-4)z^{n+4}}{} + \frac{m(m-6)z^{n+6}}{} + \ldots,$$

where $r, z$ are the running coordinates, $\varphi_{m,n}^H$ is a harmonic function satisfying the equation $\nabla^2 \varphi = 0$, $\varphi_{m,n}^{II}$ is a biharmonic function satisfying the equation (1) $\nabla^4 \varphi_{m,n} = 0$.

As an illustration, let us present several of the first polynomials

$$\varphi_{1,1} = X_1 r z, \quad \varphi_{1,2} = X_2 \left(\frac{1}{3} r z^2 - 2 z^2\right), \quad \varphi_{2,1} = X_3 \left(\frac{1}{3} r^2 z - \frac{2}{3} z^2\right),$$

$$\varphi_{2,2} = X_4 \left(\frac{1}{3} r^2 z^2 - 2 r z^2\right), \quad \varphi_{3,1} = X_5 \left(\frac{1}{3} r^3 z - \frac{8}{3} r z^2 + \frac{5}{3} z^3\right),$$

$$\varphi_{3,2} = X_6 \left(\frac{1}{3} r^3 z^2 - 2 r^2 z^2 + \frac{8}{3} z^3\right), \quad \varphi_{4,1} = X_7 \left(\frac{1}{3} r^4 z - \frac{8}{3} r^2 z^2 + \frac{16}{3} z^3\right),$$

$$\varphi_{4,2} = X_8 \left(\frac{1}{3} r^4 z^2 - 2 r^3 z^2 + 8 r z^3\right), \quad \varphi_{5,1} = X_9 \left(\frac{1}{3} r^5 z - \frac{16}{3} r^3 z^2 + \frac{64}{3} r z^3 - \frac{16}{5} z^4\right).$$
The stress components are found from the known Love formulas \[1\]

\[
\sigma_r = \frac{\partial}{\partial r} (\nu \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial r^2}), \\
\sigma_\theta = \frac{\partial}{\partial \theta} (\nu \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial \theta^2}), \\
\tau_{r\theta} = \frac{\partial}{\partial \theta} (1 - \nu) \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial \theta^2},
\]

and the displacement components from the formulas

\[
\frac{E_u}{1 + \nu} = - \frac{\partial^2 \varphi}{\partial r^2}, \\
\frac{E_w}{1 + \nu} = 2 (1 - \nu) \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2}.
\]

where \(\nu\) is Poisson's ratio, \(u\) is the horizontal and \(v\) the vertical displacement.

Using (4) and (5), we write down expressions for the stress and displacement components (at points belonging to the boundary surfaces of the cylindrical body under consideration) by limiting ourselves to stress functions with the highest terms up to \(r^4\) and \(r^4z\), inclusive, for any \(r\), \(z\), and \(v\):

\[
\sigma_r = -2 (1 - 2\nu) X_4 - 2X_3 + 16 (1 - 2\nu) zX_6 + 32zX_7 - 4 (3 - 2\nu) r^2 - \frac{2}{3} (1 - 2\nu) r^2 z \]

\[
\sigma_\theta = -2 (1 - 2\nu) X_4 - 2X_3 + 16 (1 - 2\nu) zX_6 + 32zX_7 - 4 (1 - 2\nu) \frac{r^2 z}{3} - \frac{2}{3} (1 - 2\nu) \frac{r^2 z}{3} X_7 + 432 (r^2z - 4/9z^2) X_1 - 36 (5 - 2\nu) r^4 - \frac{8}{3} (1 - 2\nu) r^4 z^3 - \frac{2}{3} (1 - 2\nu) r^4 z^3 X_7 - 30 (r^4 - 7,2r^3 z + 1, 6z^2) X_3,
\]

\[
\sigma_z = -2 (1 - 2\nu) X_4 - 2X_3 + 16 (1 - 2\nu) zX_6 + 32zX_7 - 4 (1 - 2\nu) \frac{r^2 z}{3} - \frac{2}{3} (1 - 2\nu) \frac{r^2 z}{3} X_7 + 432 (r^2z - 4/9z^2) X_1 - 36 (5 - 2\nu) r^4 - \frac{8}{3} (1 - 2\nu) r^4 z^3 - \frac{2}{3} (1 - 2\nu) r^4 z^3 X_7 - 30 (r^4 - 7,2r^3 z + 1, 6z^2) X_3,
\]

\[
\tau_{r\theta} = 0X_4 + 0X_3 + 16 (2 - \nu) rX_6 + 32rX_7 + 48 (3 - 2\nu) r^2 z X_3 + 144 (r^2 - 4rz^2) X_3 + 96 (3 - 2\nu) r^3 z X_3 + 32rX_7 + 48 (3 - 2\nu) r^2 z X_3 + 144 (r^2 - 4rz^2) X_3 + 96 (3 - 2\nu) r^3 z X_3 + 32rX_7 + 48 (3 - 2\nu) r^2 z X_3 + 144 (r^2 - 4rz^2) X_3.
\]

The expressions for the displacement components for arbitrary \(r\), \(z\), and \(v\) are

\[
\frac{E_u}{1 + \nu} = -2rX_4 - 2X_3 + 16rzX_6 + 32rzX_7 - 4 (r^3 - 2rz^2) X_8 - 4 (r^3 - 4rz^2) X_8 + 96 (1,5r^2z - 2/3rz^3) X_9 + 144 (r^3 - 4rz^2) X_3 - 6 (r^3 - 8rz^2 + \frac{8}{3} rz^3) X_7 + 36 (r^3 - 8rz^2 + \frac{8}{3} rz^3) X_7 - 30 (r^4 - 7,2r^3 z + 1, 6z^2) X_3,
\]

\[
\frac{E_w}{1 + \nu} = 8 (1 - 2\nu) rX_4 + 4zX_6 + 8 (3 - 2\nu) r^3 z X_3 + 144 (r^3 - 4rz^2) X_3 + 16 (r^2 - 2rz) X_7 + 8 (3 - 2\nu) r^3 z X_3 + 46 (2 - \nu) r^3 z X_3 + 144 (r^3 - 4rz^2) X_3 + 96 (3 - 2\nu) r^3 z X_3 + 32rX_7 + 48 (3 - 2\nu) r^2 z X_3 + 144 (r^3 - 4rz^2) X_3 + 96 (3 - 2\nu) r^3 z X_3 + 32rX_7 + 48 (3 - 2\nu) r^2 z X_3 + 144 (r^3 - 4rz^2) X_3.
\]