PLANE AND AXISYMMETRIC PROBLEM IN THE THEORY
OF LIMITING EQUILIBRIUM OF AN INHOMOGENEOUS
COHESIVE OR LOOSE MEDIUM SATISFYING
MOHR'S CONDITION

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The first investigations of the theory of limiting equilibrium (TLE) for an inhomogeneous continuum were made by Sokolovskii [1a]. Later, inhomogeneity with Coulomb's condition of limiting equilibrium was investigated by V. Ol'shak, O. Ya. Garkavi, C. Szymansky, and Z. Sobota (references to these authors can be found in [15]) and also by Sobolevskii [2], Przedeci [3], and Sobota [4], Grigor'ev [5], and Chernikov [6, 7]. Nevertheless, interest in this problem has not slackened, as shown by the appearance of new work [8-11] (Livneh and Greenstein [9] and Stroganov [11] investigated an ideally cohesive medium). In this article, in contrast with preceding ones, I study the plane and axisymmetric problems of TLE of an inhomogeneous continuum with the curvilinear envelope circles of Mohr.

1. Axisymmetric Problem. In the case of an inhomogeneous medium, it is convenient to write Mohr's condition in terms of the components of the principal stresses \( \sigma_i \) (i = 1, 2, 3):

\[
\frac{\sigma_1 - \sigma_3}{2} = \Phi \left( \frac{\sigma_1 + \sigma_3}{2} + H, k_1, k_2, \ldots, k_n \right),
\]

where \( \Phi \) is an arbitrary function of the stated arguments, and H and k_i, which are characteristics of the medium, are known functions of the coordinates r, z.

Condition (1), the two equations of equilibrium, and the Haar-Kármán condition constitute the fundamental system of equations for the axisymmetric problem of TLE in stresses [12]. Let us identify the conditions of limiting equilibrium, putting

\[
\frac{\sigma_r}{\sigma_z} = \sigma = \pm \Phi \cos 2\varphi - H; \quad \sigma = \sigma = \pm \Phi - H; \quad \tau_{rz} = \Phi \sin 2\varphi,
\]

where \( \sigma \) is the reduced mean stress \( (\sigma = \sigma_r + \sigma_z/2 + H) \) and \( \varphi \) is the angle between the first principal stress and the r axis. Then the fundamental system of equations of the antisymmetric problem of TLE in stresses can be written in the form

\[
\frac{\partial \sigma}{\partial r} \left( 1 + \frac{\partial \Phi}{\partial \varphi} \cos 2\varphi \right) + \frac{\partial \sigma}{\partial \varphi} \cos 2\varphi - 2\Phi \left( \sin 2\varphi \frac{\partial \Phi}{\partial \varphi} - \cos 2\varphi \frac{\partial \Phi}{\partial \sigma} \right) = X,
\]

\[
\frac{\partial \sigma}{\partial r} \cos 2\varphi + \frac{\partial \sigma}{\partial \varphi} \left( 1 - \frac{\partial \Phi}{\partial \varphi} \cos 2\varphi \right) + 2\Phi \left( \cos 2\varphi \frac{\partial \Phi}{\partial \varphi} + \sin 2\varphi \frac{\partial \Phi}{\partial \sigma} \right) = Y,
\]

where

\[
X = \Phi \frac{r}{r} \left( \kappa - \cos 2\varphi \right) + \frac{\partial H}{\partial r} - \sum_{i=1}^{n} \frac{\partial \Phi}{\partial k_i} \left( \frac{\partial k_i}{\partial \varphi} \cos 2\varphi + \frac{\partial k_i}{\partial \sigma} \sin 2\varphi \right),
\]

\[
Y = \gamma - \Phi \cos 2\varphi + \frac{\partial H}{\partial \varphi} - \sum_{i=1}^{n} \frac{\partial \Phi}{\partial k_i} \left( \frac{\partial k_i}{\partial \varphi} \sin 2\varphi - \frac{\partial k_i}{\partial \sigma} \cos 2\varphi \right),
\]

\( \gamma \) is the bulk density of the medium, and \( \kappa = \pm 1 \) in conformity with the Haar-Kármán condition. The equation of the characteristics of system (3) is

\[
\frac{\partial \varphi}{\partial r} = \frac{\sin 2\varphi \pm \sqrt{1 - \Phi^2}}{\cos 2\varphi + \Phi} \left( \Phi' - \frac{\partial \Phi}{\partial \sigma} \right).
\]
and therefore system (3) is of the hyperbolic, elliptic, or parabolic type according to whether $|\Phi'| < 1$, $|\Phi'| > 1$, or $|\Phi'| = 1$. Let us consider the case in which $|\Phi'| < 1$. We introduce some function $\psi$ such that
\[
\frac{\partial \Phi}{\partial \sigma} = \sin \psi. \tag{6}
\]
Then system (3) reduces to the system of equations of TLE with Coulomb’s condition [1b]. The equation of its characteristics and the relations along them can be written in the form
\[
\begin{align*}
\frac{dz}{d\sigma} &= \frac{2\Phi}{\cos \psi} \frac{d\Phi}{d\phi} = (X \mp Y \tan \psi) \frac{d\Psi}{d\sigma} \pm (X \tan \psi \mp Y) \frac{d\sigma}{dz}, \\
\frac{d\sigma}{d\phi} &= \frac{2\Phi}{\cos \psi} = \mp \frac{X \cos (\phi \pm \epsilon) - Y \sin (\phi \pm \epsilon)}{\cos \psi \cos (\phi \pm \epsilon)} \frac{d\phi}{d\sigma}
\end{align*} \tag{7}
\]
The two latter relations can be written as
\[
\frac{d\sigma}{d\phi} = \frac{2\Phi}{\cos \psi} \frac{d\phi}{d\sigma} = \frac{X \sin (\phi + \epsilon) - Y \cos (\phi + \epsilon)}{\cos \psi} \frac{d\phi}{d\sigma} \tag{8}
\]
or, introducing the derivatives of the characteristics in the directions $\alpha$ and $\beta$,
\[
\frac{\partial \sigma}{\partial \phi} \pm \frac{\partial \Phi}{\partial \psi} \frac{\partial \Phi}{\partial \phi} = \pm \frac{X \sin (\phi \pm \epsilon) - Y \cos (\phi \pm \epsilon)}{\cos \psi} \tag{9}
\]
\textbf{Note.} These considerations for the axisymmetric problem with Mohr’s condition and the Haar–Kármán postulate are easily generalized to the Mixes–Schleicher condition with the assumption that $\sigma_3 \leq \sigma_2 = \sigma_1 \leq \sigma_1$. Here we must bear in mind that a substitution of the Levi type [6d] identifying the Mises–Schleicher–Botkin condition is a particular case of the substitution
\[
\begin{align*}
\sigma_x &= \sigma \left(1 + \frac{a}{V^2} \sin \Phi \pm a \cos \Phi \cos 2\psi \right) - \frac{k}{a}; \\
\sigma_y &= \sigma \left(1 - \frac{2a}{V^2} \sin \Phi \right) - \frac{k}{a}; \\
\tau_{xy} &= \sigma a \cos \Phi \sin 2\psi,
\end{align*}
\]
where $\Phi = \omega - \pi/6$, and $\omega$ is the angle of Sokolovskii [1c] ($\tan \omega = (\sigma_1 - \sigma_2)/(\sqrt{3} (\sigma_1 - \sigma_3))$).

2. Plane Problem. It can be shown that the fundamental system of equations of the plane (plane deformation) problem in stresses of TLE for an inhomogeneous continuum with Mohr’s condition can also be reduced to system (3), in which, however,
\[
\begin{align*}
X &= \frac{\partial H}{\partial x} - \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} \left(\frac{\partial k_i}{\partial x} \cos 2\psi + \frac{\partial k_i}{\partial y} \sin 2\psi \right); \\
Y &= \gamma + \frac{\partial H}{\partial y} - \sum_{i=1}^{n} \frac{\partial \Phi}{\partial y_i} \left(\frac{\partial k_i}{\partial x} \sin 2\psi - \frac{\partial k_i}{\partial y} \cos 2\psi \right)
\end{align*} \tag{10}
\]
Here Eqs. (7)–(9) retain their form, but Eqs. (2) is to be written in the form
\[
\begin{align*}
\sigma_x &= \sigma + \Phi \cos 2\psi - H; \\
\sigma_y &= \Phi \sin 2\psi.
\end{align*} \tag{11}
\]
\textbf{Note.} The above considerations for plane deformation can also be generalized for a plane state of stress. Here, in particular, we must bear in mind that the Mises–Schleicher–Botkin condition is identified with a substitution of the form
\[
\begin{align*}
\sigma_x &= k \frac{V^2 \cos \omega \pm \sin \omega \cos 2\psi}{1 - 2 V^2 \cos \omega}; \\
\sigma_y &= k \frac{\sin \omega \sin 2\psi}{1 - 2 V^2 \cos \omega}; \\
\tau_{xy} &= \Phi \sin 2\psi.
\end{align*}
\]
Then for $\omega$ and $\psi$ we get a system of two differential equations in partial derivatives (the system is of the hyperbolic type on condition that $|2 V^2 \omega - \cos \omega| < 1$).

3. Particular Form of Equations of Characteristics. For certain forms of the Mohr envelope circles, Eqs. (7)–(9) become simpler. In the case of the plane problem, with the condition of B. D. Amin