VELOCITIES IN ZONES OF PLASTIC FLOW IN SOILS

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If soils undergoing plastic deformation can to some extent be regarded as ideally plastic bodies, the velocities in the limiting stress region can be quite strictly constructed on the basis of the plastic potential theory. The idea of applying the theory of the plastic potential to soils was first put forward by Gvozdev [1]. It was later used in papers by Drucker and Prager [2] and Shield [3]. Nevertheless, up till now the theory of the plastic potential has not received due recognition from soil mechanics specialists. One reason for this, as pointed out by Gorbunov-Posadov [5], is that the velocity fields constructed by Shield are contradicted by the experimental data. However, on close examination it was found that a mistake occurs in references [2] and [3]. Below we give a corrected solution, which agrees with the experimental data.

The following exposition deals with plane soil deformation. The flow surface is defined by the equations

\[
\Phi_1 = \pm (1 \pm \sin \varphi) \sigma_1 \mp (1 \pm \sin \varphi) \sigma_2 = 0, \tag{1}
\]

where the \( \sigma_i \) are the principal components of the stress and \( \varphi \) is the angle of internal friction of the soil.

In the case of a cohesive soil, in Eq. (1) the \( \sigma_i \) must be replaced by the reduced principal stresses \( \sigma_i + C \cot \varphi \), where \( C \) is the cohesion of the soil.

According to the theory of the plastic potential, the velocities of the principal plastic-deformation components of an isotropic body are given by the relations

\[
\dot{\sigma}_i = \dot{\Phi}_i \sigma_i, \tag{2}
\]

where \( \lambda \) is an arbitrary nonnegative factor. In Eq. (2) the dot indicates differentiation with respect to time.

By Eqs. (2) and (1), with the condition \( \sigma_1 > \sigma_2 \), we find the relation

\[
\dot{\sigma} = - \dot{\gamma}_{\text{max}} \sin \varphi, \tag{3}
\]

where \( \dot{\gamma} \) is the bulk deformation velocity, \( \dot{\gamma}_{\text{max}} \) is the velocity of maximal plastic shear.

This latter equation means that plastic deformations of the soil are accompanied by expansion at a rate directly proportional to the velocity of maximal shear: the proportionality coefficient is \( \sin \varphi \). This relation was first derived by Gvozdev.

To determine the velocity field we use Eq. (3), with the condition that the shear deformation velocity in the principal planes \( \dot{\epsilon}_{12} = 0 \), which is identically true for isotropic bodies. For small enough deformations we have

\[
\dot{v}_1 = - v_{1,1}, \quad \dot{v}_2 = - v_{2,2}, \quad \dot{v}_1 = v_{1,2} + v_{2,1},
\]

where the \( v_i \) are the components of the velocity vector along the principal axes \( y_i \), while the subscripts separated by commas indicate partial derivatives with respect to the corresponding coordinates. In the first two expressions, the minus signs are written on the principle that tensions are regarded as negative.

Thus we can write a system of equations

\[
(1 + \sin \varphi) v_{1,1} + (1 - \sin \varphi) v_{2,2} = 0, \quad v_{1,2} + v_{2,1} = 0.
\]

Analysis of this system shows that it has two real families of characteristics which coincide with the lines of sliding,

\[
dy_2 = \pm dy_1 \tan \theta \tag{5}
\]
and hence are the hyperbolic type. (For brevity we have here introduced the notation $\Phi = \pi/4 - \varphi/2$.)

Let us call the lines of sliding corresponding to the upper sign in Eq. (5) the "first family," and those corresponding to the lower sign the "second family."

Transforming to an arbitrary rectangular system of coordinates $x_i$ rotated through an angle $-\alpha$ relative to the previous system $y_i$, and taking a network of lines of sliding in the new coordinate system $s_i = s_i(x_1, x_2)$, we convert Eqs. (4) and (5) to the canonical form:

$$u_{1,s_1} + u_{2,s_1} \tan (\alpha - \delta) = 0, \quad u_{1,s_1} + u_{2,s_1} \tan (\alpha + \delta) = 0;$$

$$x_{2,s_2} = x_{1,s_1} \tan (\alpha - \theta), \quad x_{2,s_2} = x_{1,s_1} \tan (\alpha + \theta),$$

where $u_i$ are the components of the velocity vector in system $x_i$.

The first equation in Eq. (6) shows that the increment of the velocity vector is orthogonal to the first family of lines of sliding, while the second shows that the increment of the velocity vector is orthogonal to the second family of lines of sliding. In other words, Eqs. (6) express the condition of inextensibility of the slip lines. For soils, this fact was first found by Gvozdev by a different method. This property of the lines of sliding suggests that it might be convenient to transform to an expansion of the velocity vector in components $w_i$ parallel to the lines of sliding of the first and second families. The old and new components of the velocity vector are related as follows:

$$u_1 = w_1 \cos (\alpha - \theta) + w_2 \cos (\alpha + \theta);$$

$$u_2 = w_1 \sin (\alpha - \theta) + w_2 \sin (\alpha + \theta).$$

Substituting these expressions into Eq. (6), we get

$$\begin{cases}
  w_{1,s_1} + w_{2,s_1} \sin \varphi - w_2 a_{s_1} \cos \varphi = 0; \\
  w_{2,s_1} + w_{1,s_1} \sin \varphi + w_1 a_{s_2} \cos \varphi = 0.
\end{cases}$$

Let us go back and again consider a homogeneous stress field in which the lines of sliding form two families of straight lines intersecting at an acute angle $2 \delta$. The bisectrix of this angle makes an angle $\alpha = \alpha_0 = \text{const}$ with the $x_1$ axis. According to Eq. (7), the equations of the lines of sliding are

$$x_2 = x_1 \tan (\alpha_0 - \theta) + \text{const}. \quad (9)$$

Integrating Eq. (8), we get

$$w_1 + w_2 \sin \varphi = f_1(s_1), \quad w_2 + w_1 \sin \varphi = f_2(s_2),$$

where $f_i(s_i)$ are arbitrary functions of their arguments.

Hence

$$w_1 = \sec^2 \varphi f_1(s_1) - f_2(s_2) \sin \varphi, \quad w_2 = \sec^2 \varphi f_2(s_2) - f_1(s_1) \sin \varphi. \quad (10)$$

Note that in these expressions the parameter $s_1$ has a constant value along each line of sliding of the first family, and varies as we go from one line to another. In turn, parameter $s_2$ is constant along each line of the second family, and varies as we go from one line to another. By Eq. (9) we have

$$\begin{cases}
  s_1 \\
  s_2
\end{cases} = x_2 - x_1 \tan (\alpha_0 - \theta).$$

From the general Eq. (10), we can derive particular cases. Here we shall note only the very simple one (which however is very important in practice) when $w_1 = 0$ in the zone of limiting stress. Then by Eq. (10), in the same zone $w_2 = \text{const}$. This means that the region of limiting stress is displaced relative to the stationary rigid part like a rigid body in a direction parallel to the boundary line of sliding between them. The boundary line can be any line of sliding of the second family. Along this line we observe a discontinuity in the velocity vector. The fact that in this type of motion the limiting-stress region (except for an infinitely thin boundary layer) does not undergo deformation does not conflict with the theory of limiting equilibrium of the soil, based on the model of an ideal rigid-plastic soil. In this model, deformations are possible but not necessary in the limiting-stress state. This follows, if only from the fact that in Eq. (2) of the theory of the plastic potential we can put $\lambda \to 0$. In a real soil, limiting stress is unavoidably accompanied by appreciable deformations. The above case will therefore in reality mean a gradual displacement of one part of the soil, not under limiting stress, relative to another part, along some boundary slip plane.