The axisymmetrical dynamic problem of the theory of elasticity with mixed boundary conditions was studied in [1-3]. In this article it will be stated in a more general formulation. The methods of investigation will also be rather different.

Let us consider perturbed motion of an ideal elastic medium occupying the half-space $z > 0$, due to an axysymmetric displacement of a local part of the surface $z = 0$, following a given law.

This motion is characterized by a system of wave equations [1]:

\[
\begin{align*}
\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} &= \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2}; \\
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2} &= \frac{1}{b^2} \frac{\partial^2 \psi}{\partial t^2}.
\end{align*}
\]

Here $\varphi$ and $\psi$ are the elastic potentials; $c$ and $b$ are velocities of propagation of longitudinal and transverse waves,

\[
c = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}, \quad b = \sqrt{\frac{\mu}{\rho_0}};
\]

$\lambda$ and $\mu$ are the Lamé constants; and $\rho_0$ is the density of the medium.

The relations between the elastic potentials $\varphi(t, z, t)$ and $\psi(t, z, t)$, on the one hand, and the displacements $u(t, z, t)$ of points in the half-space (in the direction of the $r$ axis) and $v(t, z, t)$ (in the direction of the $z$ axis) and the normal stresses* $\sigma_{zz}(t, z, t)$ and tangential stresses $\tau(t, z, t)$, on the other, are governed by well-known equations in the theory of elasticity [4]:

\[
\begin{align*}
u &= \frac{\partial \varphi}{\partial r} - \frac{\partial \psi}{\partial z}; \\
v &= \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial r} + \frac{\psi}{r}; \\
\tau &= 2\mu \left[ \frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial z \partial r} + \frac{1}{r} \frac{\partial \psi}{\partial z} \right] + \lambda \left[ \frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right]; \\
\tau &= \mu \left[ \frac{\partial^2 \varphi}{\partial r^2} - \frac{\partial^2 \psi}{\partial z^2} + 2 \frac{\partial^2 \varphi}{\partial z \partial r} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} \right].
\end{align*}
\]

We shall take zero initial conditions. When $t = 0$

*In the future we shall omit the subscripts of the normal stresses.
In addition, we shall assume that the elastic potentials \( \varphi \) and \( \psi \) obey the following boundary conditions:

\[
\begin{align*}
\varphi & \to 0; \quad \psi \to 0 \\
z & \to \infty \quad z \to \infty \\
r \varphi I_0 (r \xi) & \to 0; \quad r \psi I_1 (r \psi) \to 0 \\
r & \to 0 \\
r & \to \infty
\end{align*}
\]

To solve the problem we shall use integral transformations. Following [2, 5, 6], let us apply the first- and second-order Hamel transformations in the coordinate \( r \) and the Laplace transformation in the time \( t \) to the system (1)-(2). Then we get

\[
\begin{align*}
\frac{d^2 \varphi}{d z^2} - \left( \frac{\xi^2}{c^2} + \frac{s^2}{b^2} \right) \varphi &= 0; \\
\frac{d^2 \psi}{d z^2} - \left( \frac{\xi^2}{c^2} + \frac{s^2}{b^2} \right) \psi &= 0.
\end{align*}
\]

Here,

\[
\begin{align*}
\varphi (\xi, s, z) &= \int_0^\infty \tilde{\varphi} (r, s, z) I_0 (r \xi) r dr; \quad \tilde{\varphi} (r, s, z) = \varphi (r, t, z); \\
\psi (\xi, s, z) &= \int_0^\infty \tilde{\psi} (r, s, z) I_1 (r \xi) r dr; \quad \tilde{\psi} (r, s, z) = \psi (r, t, z).
\end{align*}
\]

The solutions to (10) and (11) with conditions (8) and (9) are

\[
\begin{align*}
\varphi (\xi, s, z) &= A (\xi, s) \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right); \\
\psi (\xi, s, z) &= B (\xi, s) \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right).
\end{align*}
\]

On the basis of (13), (14), and (4)-(7), the transformants of the displacements and stresses are expressed in terms of the functions \( A(\xi, s) \), \( B(\xi, s) \) as follows:

\[
\begin{align*}
\overline{v} (\xi, s, z) &= \mu \left[ 2 \xi A (\xi, s) \sqrt{\xi^2 + s^2/b^2} \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right) \\
&- B (\xi, s) \left( 2 \xi^2 + s^2/b^2 \right) \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right) \right]; \\
\sigma (\xi, s, z) &= \mu \left[ A (\xi, s) \left( s^2/b^2 + 2 \xi^2 \right) \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right) \\
&- B (\xi, s) 2 \xi \sqrt{\xi^2 + s^2/b^2} \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right) \right]; \\
\overline{u} (\xi, s, z) &= -\xi A (\xi, s) \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right) + B (\xi, s) \sqrt{\xi^2 + s^2/b^2} \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right); \\
\overline{\sigma} (\xi, s, z) &= -\sqrt{\xi^2 + s^2/b^2} A (\xi, s) \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right) + \xi B (\xi, s) \exp \left( -z \sqrt{\xi^2 + s^2/b^2} \right).
\end{align*}
\]

Putting \( z = 0 \), we determine the functions \( A(\xi, s) \) and \( B(\xi, s) \) in terms of the transformants of the stresses:

\[
A (\xi, s) = \frac{1}{\mu} \tau (\xi, s) \left( 2 \xi^2 + s^2/b^2 \right) - \frac{1}{\mu} \overline{\tau} (\xi, s) \sqrt{\xi^2 + s^2/b^2} \\
B (\xi, s) = \frac{1}{\mu} \overline{\sigma} (\xi, s) 2 \xi \sqrt{\xi^2 + s^2/b^2} - \frac{1}{\mu} \tau (\xi, s) \left( s^2/b^2 + 2 \xi^2 \right). 
\]