METHOD FOR \textit{a priori} COMPUTATION OF MEASUREMENT ACCURACY IN SYSTEMS WITH SYSTEMATIC ERROR

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Using the general statement of the problem, we consider determination of the accuracy of direct measurements related to indirect measurements by a known relationship in which the systematic error is estimated during processing of experimental data.

For indirect measurements using a known relationship between the desired quantities $x_j$, $j = 1, 2, ..., m$, and directly measured quantities $l_i$, $i = 1, 2, ..., n \geq m$, the task of choosing the means of measuring $l_i$ raises the problem of determining the accuracy with which we must measure these quantities in order to evaluate the quantities $x_j$ with the accuracy required to meet the requirements of the experiment. The general version of this problem was considered in [1], which presented an analytic solution for models of measurement results $l_i$ that do not contain systematic errors $c_i$. Such a model can be used when the systematic error can be determined by calibrating apparatus and then eliminating the error from the measurement results. If, for some reason, we cannot do this, for measurement systems provided with the so-called property of structural measurement redundancy, which estimate the systematic error from the results of processing of experimental data [2] with $n > m$, our problem of determining measurement accuracy can be solved by using the results of [3]. The condition stated in [3] for satisfaction of measurement-accuracy requirements for this class of systems was introduced under the assumption that a linear approximation of the true values of the measured quantities $l_i$ is described by a two-component regression model, and the estimates $\hat{c}_i$ found as estimates of the system error $c_i$ and the estimates $\hat{x}_j$ for the values of $x_j$ are statistically independent. Nonetheless, practical considerations that arise when we attempt to obtain unbiased estimates of $\hat{x}_j$ from measurement results $l_i$ make it necessary for the initial regression model to include additional terms and to take into account the correlations between the random variables $c_i$ and $x_j$, $i = 1, 2, ..., n, j = 1, 2, ..., m$.

Below we consider the problem of determining (calculating) measurement accuracy by using the conditions of [1, 3] and the additional assumption that the measurement results $l_i$, $i = 1, 2, ..., n \geq m$, are described by a regression model with an arbitrary number of components, where the estimates for the system error $\hat{c}_i$ and the results of indirect measurement $\hat{x}_j$ are statistically dependent, since they are found from the results of measurements of the same quantities $l_i$, $i = 1, 2, ..., n$.

Our scheme for solution of this problem, which provides a method for computing measurement accuracy in systems with systematic error, consists of the following stages:

1. In the first stage we establish functional relations between the measured variables $l_i$, $i = 1, 2, ..., n$, and the variables $x_j$, $j = 1, 2, ..., m < n$, to be determined. They are composed in such a way that the measurement system has the properties of structural redundancy formulated, in particular, in [2].

2. We use results of analysis of the properties of the measurement facilities in the system to construct a model of the measurements that describes the functional relationships between the quantities $x_j$, $l_i$, and the properties of the errors in the system.

3. We use the model and measurement results $l_i$, $i = 1, 2, ..., n$, to find an estimate $\hat{c}_i$ for the systematic error $c_i$, $i = 1, 2, ..., n$. This estimate is then eliminated (subtracted) from the measurement results and we use the “measurement results” thus obtained to compute the estimates $\hat{x}_j$, $j = 1, 2, ..., m$.

4. We specify quantities (parameters) characterizing the accuracy required of the estimates $\hat{x}_j$.

5. We establish relations between the quantities (parameters) characterizing the accuracy of the results of direct measurement and the required accuracy in the results of estimation $\hat{x}_j$. 

Translated from Izmeritel'naya Tekhnika, No. 11, pp. 9-14, November, 1997.
6. Using the relationships we have found and the specified values for the quantities characterizing the required accuracy for the estimates \( \tilde{x}_j \), we compute the values of parameters characterizing the accuracy required from the results of direct measurement \( \tilde{I}_i \) for the estimates \( \tilde{x}_j \) to have the required accuracy.

This stage completes the solution of the problem of determining the measurement accuracy in a system subject to systematic error. The values found for the quantities characterizing the accuracy of \( \tilde{I}_i \) are the results of solution.

We now turn to a mathematical formulation and solution of the problem under investigation with the scheme we have described.

We state the initial data (the conditions \( \Omega \)) as follows.

\( \Omega : \) Suppose we are given an equation \( \tilde{L} = L + C + \Xi \) that relates the vector of measurement results \( \tilde{L} \) with the true values of the measured quantity \( L \) and the systematic error \( C = \sum_{h=1}^{H} G_h B_h \) and the random error \( \Xi \) of the measurements.

As a rule, the relationship between the measured values and the \( X \) determined from them is nonlinear. As a result, we estimate \( X \) with the method of successive approximation. For this purpose, in the step \( \omega \) we construct a regression equation connecting the measurement results \( \tilde{L} \) and the desired vector quantity \( X \):

\[
\tilde{L} = L(X^{(\omega)}) + A^{(\omega)} \Delta X^{(\omega)} + \sum_{h=1}^{H} G_h B_h + \Xi.
\]

In (1) we have used the following notation.

\( L(X^{(\omega)}) = L^{(\omega)} \) is the value of the measured variable \( L \) as computed in the \( \omega \)-th step of the computation from the relationship \( L(X) \) for fixed (given \textit{a priori}) value of \( X = X^{(\omega)} \);

\( A^{(\omega)} = (\partial L / \partial X)L^{(\omega)}L^{(\omega)} \) is the matrix of partial derivatives as computed from the \textit{a priori} specified values \( L^{(\omega)} \);

\( \Delta X^{(\omega)} \) is the increment in \( X \) as computed in the \( \omega \)-th step of the computation;

\( L \) is the \( n \times n \) block vector composed of the \( n \)-dimensional blocks \( L_k : nL = [L_1 | L_2 | ... | L_k | L_n]^T \), where \( T \) denotes transposition. Each of the blocks \( L_k \), \( k = 1, 2, ..., s \), is composed of the values \( l_{ik} \), \( i = 1, 2, ..., n \) taken at the fixed times \( t_k \). Thus, \( \tilde{L}_k = [l_{1k} | l_{2k} | ... | l_{ik} | ... | l_{nk}]^T \). The block vector \( \tilde{L} \) has an analogous structure, being composed of the measurement results \( \tilde{l}_{ik} \);

\( X \) is the \( m \times n \) block vector composed of the quantities \( x_{jk} = x_j(t_k) \), \( j = 1, 2, ..., m \), computed from the results of measurements \( \tilde{L} \) at time \( t_k \). It has the following structure: \( X = [X_1 | X_2 | ... | X_k | ... | X_m]^T \), where \( X_k = [x_{1k} | x_{2k} | ... | x_{jk} | ... | x_{mk}]^T \). The block vector \( \tilde{X} \) has an analogous structure;

\( A = \partial L / \partial X \) is a known \( n \times m \) matrix of partial derivatives used for the linear approximation \( \tilde{L} = AX \). It has the following block structure:

\[
A = \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_s
\end{bmatrix},
\]

\[
A_k = \begin{bmatrix}
a_{i1}^{(k)} & \cdots & a_{in}^{(k)} \\
\vdots & \ddots & \vdots \\
a_{n1}^{(k)} & \cdots & a_{nn}^{(k)}
\end{bmatrix} = \frac{\partial l_{ik}}{\partial x_{jk}}.
\]

(2)

\( G_h = [G_{h1} | G_{h2} | ... | G_{hk} | ... | G_{hn}]^T \) is an \( n \times n_P \) block matrix; \( B_h = [\theta_h^{(1)} | \theta_h^{(2)} | ... | \theta_h^{(H)} | ... | \theta_h^{(n_P)}]^T \) is an \( n_P \)-dimensional block vector; these are introduced for analytic description of the systematic error \( C = \sum_{h=1}^{H} G_h B_h \). The blocks that appear in them have the following structure:

\[
G_{hk} = \begin{bmatrix}
G_{(1)}^{(h)} & 0 & \cdots & 0 \\
0 & G_{(2)}^{(h)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & G_{(n)}^{(h)}
\end{bmatrix},
\]

\[
G_{hk}^{(j)} = \begin{bmatrix}
G_{hk1}^{(j)} & G_{hk2}^{(j)} & \cdots & G_{hkp}^{(j)}
\end{bmatrix}.
\]

(3)