Two mathematical models of formation of the normal probability density of random measurements errors are considered. The models are based on an application of partial differential equations of high orders.

It is indeed difficult to pinpoint an area of mathematical processing of arbitrary observations where a solution of the problems arising therein would not have been associated with the normal probability distribution.

It should be noted that the probability integral appears as a constituent part in the proof of the law of large numbers, in the theory of infinitely divisible distribution laws, in the theory of stochastic processes and many other scientific directions.

Specialists in a wide variety of fields associate their scientific investigations with the law of large numbers in a hope to obtain new lawlike relations in phenomena under study.

Every science presupposes an experiment with stable results, i.e. the phenomenon under consideration always occurs (or does not occur).

What is then the meaning of statistical stability? It should be admitted that at present there is no fully satisfactory answer to this question. In the classical literature certain requirements on statistical stability are known, formulated by von Mises earlier in this century [1]. Nevertheless, there is a definite "no" in the notion \( n \to \infty \), \( n \) is the number of observations.

In astronomical-geodesical practice, random errors occur as a result of a large number of independent fluctuations which, with probability arbitrarily close to 1, can be considered to be elementary random components of a general random measurement error. The observed random error \( \Delta \) can be formed in a rather complicated manner as a result of interaction of random fluctuations \( \delta_i \). If one traces the dependence of their total effect on \( \Delta \), it is then represented in a form of some structurally complicated function

\[
\Delta = \Psi(\delta_1, \delta_2, \ldots, \delta_n).
\]

Assuming a smooth dependence of the function on their arguments, one could utilize for infinitely small values of the arguments a Taylor expansion in the form

\[
\Psi(\delta_1, \delta_2, \ldots, \delta_n) = a_0 + \sum_{i=1}^{n} a_i \delta_i,
\]

where \( a_0 = \Psi(0, 0, \ldots, 0) \), \( a_i = \partial \Psi / \partial \delta_i \) for \( \delta_1 = \delta_2 = \cdots = \delta_n = 0 \).

The proposed interpretation allows us – for the case of small effects \( \delta_i \) – to assume at least in a first approximation that the total effect is a linear function. Such an assumption is quite satisfactory and has been completely approbated based on the results of numerous engineering-geodisic measurements.
P. S. Laplace and C. F. Gauss — who are the founding fathers of the theory of errors in measurements and the method of the least squares — were well aware of this fact, and independently arrived at the distribution law of random variables which is now called the Laplace–Gauss law.

As it is known, the principles of summation of independent random variables were presented by J. Bernoulli in his book *Ars Conjectandi* which was published in Basel in 1713. In this book the first proof of the limit theorem of the probability theory is presented. The history of mathematical probability theory contains quite a few important names which contributed to the law of large numbers. It is of interest to note that the central limit theorem of this law provides an unexpected and remarkable result, since out of disorder in probabilities, a curve originates somehow which is determined by a well-known expression of the probability density of a normal distribution.

In practice of geodesical measurements, we have a number of elementary random errors $\delta_1, \delta_2, ...$ normally distributed with the parameters $(a, \sigma^2)$. In accordance with the law of large numbers, the following inequality is valid:

$$P\left\{ \left| \frac{\delta_1 + \delta_2 + ... + \delta_n}{n} - a \right| > \varepsilon \right\} \to 0.$$  

On the basis of the central limit theorem we obtain

$$P\left\{ \left| \frac{\delta_1 + \delta_2 + ... + \delta_n}{n} - a \right| > \varepsilon \right\} = 2\Phi\left( \frac{-\varepsilon \sqrt{n}}{\sigma} \right).  \tag{1}$$

If $\varepsilon = 3\sigma/\sqrt{n}$, then (1) will be equal to 0.997. Note that $\varepsilon$ decreases as $(1/n)^{1/2}$. Thus, replacing the actual value of the resulting error by its arithmetic mean $[\delta]/n$, we arrive at the error which is inversely proportional to $n^{1/2}$.

In practice of statistical analysis of geodesic measurements, the problem arises of estimating the error when one replaces the probability of the corresponding frequency with a definite reliability. In this case, proceeding from the Bernoulli theorem, the parameter $\varepsilon_\alpha$ is determined from the relation

$$2\Phi\left( \frac{-\varepsilon_\alpha \sqrt{n}}{m \left( \frac{1-m}{n} \right)} \right) \leq \alpha,$$

where $\alpha$ is the reliability level.

We shall consider one of the several mathematical models of formation of the probability density of measurement errors utilizing partial differential equations.

We shall postulate the following propositions: a measurement error is a result of a large number of elementary components of hierarchical structure; formation of the measurement errors is so irregular that the interaction of the elementary components can be described only by means of probabilistic principles under the assumption of the independence hypothesis [3].

Assume that at a certain instant of time, as the result of unfavorable interaction of elementary errors, destabilization of a measurement occurs, so that the trajectory of a measurement departs from the assigned level and the tendency towards stratification of a sample is beginning to show. Then a mixture of distributions with densities $f_1(x, n_1, a, \sigma^2)$ and $f_2(x, n_2, a + \lambda \sigma, \sigma^2)$, where $(a, \sigma)$ are the parameters of the distributions and $\lambda$ is the shift parameter of the center of the distribution of one population relative to the center of the basic population is now being subject to an analysis. However, in this model one ought to take into account the following very important assumption: destabilization of a measurement does not occur instantaneously, and in the mechanism of formation of the sample $f_2$ random variables belonging to the sample $f_1$ occur, which are the source of attraction of anomalous variables.

When $\lambda \leq 3$, a measurement is assumed to be stable and large errors do not affect substantially the parameters of the distribution (they sort of dissolve in the body of the measurement information) and thus $f_2$ does not depend on the location of the mathematical expectation on the coordinate axis. An American statistician, John Tukey, concluded in his investigation that,