POLYNOMIAL SOLUTIONS TO PIEZOELECTRIC BEAMS (II)  
——ANALYTICAL SOLUTIONS TO TYPICAL PROBLEMS*  

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Abstract: For the orthotropic piezoelectric plane problem, a series of piezoelectric 
beams is solved and the corresponding analytical solutions are obtained with the trial- 
and-error method on the basis of the general solution in the case of three distinct 
eigenvalues, in which all displacements, electrical potential, stresses and electrical 
displacements are expressed by three displacement functions in terms of harmonic 
polynomials. These problems are cantilever beam with cross force and point charge at 
free end, cantilever beam and simply-supported beam subjected to uniform loads on the 
upper and lower surfaces, and cantilever beam subjected to linear electrical potential.  
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Introduction  

This paper is a continuation of Ref. [1], in which a series of orthotropic piezoelectric 
plane problems was solved and the corresponding exact solutions were obtained with the trial- 
and-error method, on the basis of the general solution expressed by three displacement 
functions in terms of harmonic polynomials. By means of those simple and typical exact 
solutions and the method used in Ref. [1], one can conveniently derive the analytical 
solutions for simply supported piezoelectric beams and cantilever beams subjected to several 
kinds of typical loads and electrical potential by the principle of superposition. Based on 
these typical analytical solutions, we can clearly find out the distributing laws of 
displacements, electrical potentials, stresses and electric displacements along the height and 
length of piezoelectric beam directions. So, the analytical solutions can not only provide

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references for developing theory of practical engineering piezoelectric beam, but also serving as benchmarks and examples for numerical methods together with those exact solutions presented in Ref. [1].

1 Cantilever Beam with Cross Force P and Point Charge Q at Free End

Using $\psi(x, z)$ and $\phi(x, z)$ in Eq. (B2) in Appendix B of Ref. [1], we constitute the displacement function:

$$\psi_j = B_{ij} xz_j + B_{ij} (x^2 z_j - xz_j)$$  \hspace{1cm} (j = 1, \cdots, 3), \hspace{1cm} (1)

where $B_{ij}$ and $B_{ij}$ are unknown constants to be determined.

Substituting Eq. (1) into Eq. (3) of Ref. [1] and superposing the rigid body displacement in z-direction and identical potential solution Eq. (11) of Ref. [1] results in

$$u = \sum_{j=1}^{3} B_{ij} xz_j + \sum_{j=1}^{3} B_{ij} (3x^2 z_j - xz_j^2),$$

$$w = w_0 + \sum_{j=1}^{3} s_j k_{ij} B_{ij} x + \sum_{j=1}^{3} s_j k_{ij} B_{ij} (x^3 - 3xz_j^2),$$

$$\Phi = \Phi_0 + \sum_{j=1}^{3} s_j k_{ij} B_{ij} x + \sum_{j=1}^{3} s_j k_{ij} B_{ij} (x^3 - 3xz_j^2),$$

$$\sigma_x = -6 \sum_{j=1}^{3} \omega_{ij} B_{ij} xz_j, \hspace{1cm} \sigma_z = -6 \sum_{j=1}^{3} \omega_{ij} B_{ij} xz_j,$$

$$\tau_{zz} = \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} + \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} (3x^2 - 3z_j^2),$$

$$D_x = \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} + \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} (3x^2 - 3z_j^2), \hspace{1cm} D_z = -6 \sum_{j=1}^{3} \omega_{ij} B_{ij} xz_j.$$

The boundary conditions of a cantilever piezoelectric beam with a cross force $P$ and point charge $Q$ at free end ($x = 0$) can be expressed as follows:

$$z = \pm h/2, \hspace{1cm} \sigma_z = 0, \hspace{1cm} \tau_{zz} = 0, \hspace{1cm} D_x = 0,$$

$$x = 0, \hspace{1cm} \sigma_x = 0, \hspace{1cm} \int_{-h/2}^{+h/2} \tau_{xx} dz = Q_1, \hspace{1cm} \int_{-h/2}^{+h/2} D_x dz = Q_2,$$

$$x = L, \hspace{1cm} z = 0, \hspace{1cm} u = 0, \hspace{1cm} w = 0, \hspace{1cm} \frac{\partial w}{\partial x} = 0.$$  \hspace{1cm} (3c)

where $Q_1 = -P, Q_2 = Q$.

Substituting Eq. (2) into Eq. (3), yields

$$\sum_{j=1}^{3} s_j \omega_{ij} B_{ij} = 0, \hspace{1cm} \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} - \frac{3h^2}{4} \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} = 0,$$

$$\sum_{j=1}^{3} s_j k_{ij} B_{ij} = 0, \hspace{1cm} \sum_{j=1}^{3} s_j k_{ij} B_{ij} + 3L^2 \sum_{j=1}^{3} s_j k_{ij} B_{ij} = 0,$$

$$h \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} - \frac{h^3}{4} \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} = Q_1, \hspace{1cm} h \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} - \frac{h^3}{4} \sum_{j=1}^{3} s_j \omega_{ij} B_{ij} = Q_2,$$

$$w_0 + L \sum_{j=1}^{3} s_j k_{ij} B_{ij} + L^3 \sum_{j=1}^{3} s_j k_{ij} B_{ij} = 0.$$  \hspace{1cm} (7)