Correlations Between Zeros of a Random Polynomial

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We obtain exact analytical expressions for correlations between real zeros of the Kac random polynomial. We show that the zeros in the interval (-1, 1) are asymptotically independent of the zeros outside of this interval, and that the straightened zeros have the same limit-translation-invariant correlations. Then we calculate the correlations between the straightened zeros of the O(1) random polynomial.

KEY WORDS: Real random polynomials; correlations between zeros; scaling limit; determinants of block matrices.

1. INTRODUCTION

Let \( f_n(t) \) be a real random polynomial of degree \( n \),

\[
f_n(t) = c_0 + c_1 t + \cdots + c_n t^n
\]

where \( c_0, c_1, \ldots, c_n \) are independent real random variables. Distribution of zeros for various classes of random polynomials is studied in the classical papers by Bloch and Polya [BP], Littlewood and Offord [LO], Erdős and Offord [EO], Erdős and Turán [ET], and Kac [K1-K3]. We will assume that the coefficients \( c_0, c_1, \ldots, c_n \) are normally distributed with

\[
E c_j = 0, \quad E c_j^2 = \sigma_j^2
\]

In the case when

\[
\sigma_j^2 = 1
\]
$f_n(t)$ is the Kac random polynomial. Another interesting case is when

$$\sigma_j^n = \binom{n}{j}$$

As is pointed out by Edelman and Kostlan [EK], "this particular random polynomial is probably the more natural definition of a random polynomial." We call this polynomial the $O(1)$ random polynomial because its $m$-point joint probability distribution of zeros is $O(1)$-invariant for all $m$ (see Section 5 below). The $O(1)$ random polynomial can be viewed as the Majorana spin state [Maj] with real random coefficients, and it models a chaotic spin wavefunction in the Majorana representation. See the papers by Leboeuf [Lebl, Leb2], Leboeuf and Shukla [LS], Bogomolny, Bohigas, and Leboeuf [BBL2], and Hannay [Han], where the SU(2) and some other random polynomials are introduced and studied, that represent the Majorana spin states with complex random coefficients.

Let \{\tau_1, \ldots, \tau_k\} be the set of real zeros of $f_n(t)$. Consider the distribution function of the real zeros,

$$P_n(t) = \text{E} \# \{ j : \tau_j \leq t \}$$

where the mathematical expectation is taken with respect to the joint distribution of the coefficients $c_0, \ldots, c_n$. Let

$$p_n(t) = P_n'(t)$$

be the density function. By the Kac formula (see, e.g., [K3]),

$$p_n(t) = \frac{\sqrt{A_n(t) C_n(t) - B_n^2(t)}}{\pi A_n(t)}$$

(1.3)

where

$$A_n(t) = \sum_{j=0}^{n} \sigma_j^n t^{2j}$$

$$B_n(t) = \sum_{j=1}^{n} \frac{j \sigma_j^n t^{2j-1}}{2} = \frac{A_n'(t)}{2}$$

$$C_n(t) = \sum_{j=1}^{n} j^2 \sigma_j^n t^{2j-2} = \frac{A_n''(t)}{4} + \frac{A_n'(t)}{4t}$$

(1.4)

The derivation of (1.3) by Kac is rather complex. A short proof of (1.3) is given in the paper [EK] by Edelman and Kostlan. See also the papers by