TESTING OF VARIANCE HYPOTHESES FOR
NONSTATIONARY WHITE GAUSSIAN NOISE

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Testing of hypotheses as to the equality and proportionality of noise variance to specified time functions is discussed.

Analysis of noise in radio communications channels is of very great importance in the design of HF and UHF radio communication systems.

White Gaussian noise, in which noise measurements \( x_i \) made at certain times are normally distributed independent random variables with zero mean value \( M(x_i) = 0 \), offers a fairly good mathematical model of noise of this kind. White Gaussian noise is usually understood as noise in which the variances of all \( x_i \) are equal, i.e., \( D(x_i) = \sigma^2 \). However, the noise encountered in practice is usually not stationary, i.e., the variances \( D(x_i) = \sigma^2_i \) vary in time. We shall henceforth refer to such a process as nonstationary white noise.

Let the noise measurements be made at equal time intervals, and let \( \{x_1, x_2, \ldots, x_N\} \) be the measured values. Below we study a mathematical model of the noise in which the joint probability density of the \( \{x_i\} \) has the form

\[
p(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N} \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left(-\frac{x_i^2}{2\sigma_i^2}\right),
\]

and refer to it as nonstationary white Gaussian noise.

Many problems arise in study of such processes. In this paper we shall discuss problems associated with testing of hypotheses as to the behavior of the noise variance \( \sigma^2_i \) as a function of \( i \), i.e., actually as a function of time.

TESTING THE HYPOTHESIS THAT THE VARIANCE IS EQUAL TO A GIVEN FUNCTION

Consider testing of a statistical hypothesis of the form:

\[
H_0 : D(x_i) = f(i), \quad i = 1, N,
\]

\[
H_1 : D(x_i) \neq f(i).
\]

In other words, test whether the dependence of \( D(x_i) \) on \( i \) has the specified form \( f(i) \).

To test this hypothesis, we construct a statistic that embodies the following basic idea: find a statistic \( S \) whose mean value is equal to zero under hypothesis \( H_0 \) but always greater than zero under the alternative \( H_1 \). It is desirable that the variance of this statistic be equal to 1 at least in the asymptotic case \( N \to \infty \). Then the following decision rule will appear quite natural: if \( S < C \), where \( C \) is a certain threshold constant, adopt hypothesis \( H_0 \), but if \( S \geq C \), adopt the alternative \( H_1 \).

Let us describe the basic steps in construction of this statistic. Since the hypothesis \( H_0: D(x_i) = f(i) \) is to be verified, it is natural to take a statistic of the form

\[ S = \sum_{i=1}^{N} \left( \frac{x_i}{\sigma_i} \right)^2 - N. \]

\[ S_1 = \frac{1}{N} \sum_{i=1}^{N} (x_i^2 - \bar{f}(i))^2. \]  

(3)

Let the true variance be \( D\{x_i\} = f(i) \), where \( f(i) \) is a certain arbitrary function. We find the mean value of statistic \( S_1 \).

Expanding the square and recognizing that \( M\{x_i^2\} = f(i) \), \( M\{x_i^4\} = 3f^2(i) \) for Gaussian random variables, we obtain

\[
M[S_1 | H_1] = \sum_{i=1}^{N} (f_i(i) - \bar{f}(i))^2 + 2 \sum_{i=1}^{N} f_i^2(i).
\]

(4)

We see that the first term in (4) has the required property, since it vanishes when \( f_i(i) = \bar{f}(i) \) and is always positive when \( f_i(i) \neq \bar{f}(i) \). However, the second term in (4) is redundant and must be removed. Since this term owes its appearance to the coefficient 3 in \( M\{x_i^4\} \), it will disappear if this coefficient becomes equal to 1. Therefore the corrected statistic has the form

\[
S_2 = \frac{1}{3} \sum_{i=1}^{N} x_i^2 - 2 \sum_{i=1}^{N} x_i f(i) + \sum_{i=1}^{N} f^2(i).
\]

(5)

Now evaluation of the mean value of this statistic gives

\[
M[S_2 | H_1] = \sum_{i=1}^{N} (f(i) - f(i))^2,
\]

(6)

from which we see that \( M[S_2 | H_0] = 0; M[S_2 | H_1] > 0 \), i.e., this statistic meets the required condition.

To obtain the final form of this statistic we calculate its variance under hypothesis \( H_0 \), when \( D\{x_i\} = f(i) \). Remembering that the \( x_i \) are independent, we obtain

\[
D(S_2 | H_0) = \frac{8}{3} \sum_{i=1}^{N} f^4(i).
\]

(7)

For practical application it is advantageous to normalize \( S_2 \) in such a way that its variance will be 1 under hypothesis \( H_0 \).
Therefore the final form of statistic \( S \) in testing the hypothesis that the variance is equal to the given function \( f(i) \) should be

\[
S = \sqrt{\frac{8}{3} \sum_{i=1}^{N} f^4(i)} \left( \frac{1}{N} \sum_{i=1}^{N} \left( x_i^2 - 2x_i f(i) + f^2(i) \right) \right).
\]

(8)

The following properties will be satisfied for it:

\[
M[S | H_0] = 0; \quad D[S | H_0] = 1;
\]

\[
M(S | H_1) = \sum_{i=1}^{N} (f(i) - f(i))^2 \left( \frac{1}{N} \sum_{i=1}^{N} f^4(i) \right) > 0.
\]

(9)

The following theorem states the problem of choosing the threshold constant \( C \) in the decision rule:

**Theorem 1.** If for a certain \( \delta > 0 \)

\[
\lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{i=1}^{N} f^{4+2\delta}(i) \left( \frac{1}{N} \sum_{i=1}^{N} f^4(i) \right)^{1+\frac{3}{2}} \right\} = 0,
\]

(10)

then as \( N \to \infty \) under hypothesis \( H_0 \), \( S \) converges across the distribution to \( N(0, 1) \), i.e., to a normal random variable with zero mean value and unit variance.

Without presenting the proof of this theorem, we note only that it reduces to proof of the conditions of the Lyapunov theorem [1].

We can now state the final decision rule, at least for the asymptotic case \( N \to \infty \).