SPONTANEOUS PARAMETRIC TRANSITION FROM PERIODIC MOTION TO CHAOS IN A DYNAMICAL SYSTEM WITH TWO DEGREES OF FREEDOM

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A study is made of a spontaneous parametric transition to chaos by the disruption of periodic motion in a generating structure with two degrees of freedom. A comparison of the properties of the system examined in this study and the properties of objects with an infinite-dimensional phase space shows the universality of the spontaneous parametric scenario for a loss of stability of periodic motion that leads to randomization of the oscillatory process.

The cumulative body of research into the origin of chaotic types of oscillations in dynamical systems has established that, in contrast to that which was assumed earlier, complex finite aperiodic motions are not characteristic only of objects with a substantial number of degrees of freedom. In addition, the requirements on the number of dimensions that the phase space of systems exhibiting dynamic chaos must have has been reduced to a liminally small value, equal to three [2, 3].

Problems on the loss of stability of periodic motion are becoming very important in many cases of practical significance. The interest in this subject is twofold in character and stems from a desire to either suppress aperiodic oscillations or to excite such oscillations in order to make practical use of them. A spontaneous parametric mechanism of loss of stability of periodic motion and the subsequent generation of random oscillations was studied in [4, 5] in a special class of dynamical systems with an infinite-dimensional phase space — in self-maintained circuits with a lag. The results obtained in these studies were used to formulate sufficient conditions for realization of the same mechanism in dynamical systems with a finite number of dimensions. The goal of the present investigation is to generalize the universality of the spontaneous parametric method of changing over to random types of oscillations. The problem is solved by constructing a mathematical model of the dynamical system which has just two degrees of freedom. However, it is proven that randomization of motion in such a system occurs in the same manner as in infinite-dimensional systems.

In accordance with the findings in [4, 5], several requirements must be satisfied for spontaneous parametric randomization of motion to occur in a certain dynamical system. The system must exhibit nonlinearity characterized by ascending and descending branches. The positive curvature of the ascending branch must be great enough to ensure smooth spontaneous excitation of the fundamental mode with the frequency \( \omega_0 \), which corresponds to zero phase shift in the feedback loop. The descending branch is necessary to introduce an additional phase shift equal to \( \pi \) at the moments of time when the signal of the fundamental mode approaches the region of negative curvature of nonlinearity as the total amplitude increases. If the linear part of the system in this case turns out to be such as to ensure excitation of the second mode at the frequency \( \omega_1 > \omega_0 \), with the phase shift being equal to \( -\pi \) in this mode prior to nearing the region of negative curvature, then the necessary conditions will be created for the interaction of two free oscillations of different frequencies over a fraction of the period \( T_0 = 2\pi/\omega_0 \). These conditions may be sufficient to generate dynamic chaos if the descending branch of the nonlinear characteristic has a certain curvature [4, 5].

The above requirements can be satisfied by an oscillatory system consisting of the closed feedback loop of a nonlinear amplifier and a closed circuit formed by three series-connected electric filters. The connection between the output and the input in the feedback loop is described by the function \( F(x) \). The three filters correspond to linear differential operators \( K_1(p), K_2(p), K_3(p), p = d/dt \).

We designate the signal at the input of the nonlinear amplifier as \( x_1(t) \), while \( x_2(t), x_3(t) \) represent the signals at the outputs of the first and second filters, respectively. Then the given circular system can be described by...
the following system of differential equations:

\[
\begin{align*}
  x_1(t) &= K_3(p)x_3(t), \\
  x_2(t) &= K_1(p)F[x_1(t)], \\
  x_3(t) &= K_2(p)x_2(t).
\end{align*}
\] (1)

Let us specify the form of the linear and nonlinear operators in (1). We approximate the type of nonlinearity by using the relation proposed in [6]

\[F(x) = kx \exp(-x^2).\] (2)

This relation has both ascending and descending branches and, in contrast to the widely used cubic polynomial, remains finite for any value of the argument.

As the first filter, we choose a first-order low-frequency RC-filter with the time constant \( \tau_1 \). Then the corresponding linear differential operator \( K_1(p) \), amplitude-frequency characteristic \( K_1(\omega) \), and phase-frequency characteristic \( \varphi_1(\omega) \) (AFC and PFC), respectively, are determined in the form

\[K_1(p) = (\tau_1p + 1)^{-1},\] (3)

\[K_1(\omega) = \left(\frac{1}{\sqrt{1 + (\omega \tau_1)^2}}\right)^{-1}, \quad \varphi_1(\omega) = -\arctan \omega \tau_1.\] (3a)

As the second filter, we choose an oscillatory RCL-circuit with a normalized resonance frequency equal to unity and the passband \( \varepsilon \). Then

\[K_2(p) = (p^2 + \varepsilon p + 1)^{-1},\] (4)

\[K_2(\omega) = \frac{1}{\sqrt{1 - \omega^2}} \frac{1}{1 + (\varepsilon \omega)^2}, \quad \varphi_2(\omega) = -\arctan \frac{\varepsilon \omega}{1 - \omega^2}.\] (4a)

As the third frequency-dependent element, we choose a first-order high-frequency RC-filter with the time constant \( \tau_2 \):

\[K_3(p) = \frac{\tau_2 p}{\tau_2 p + 1},\] (5)

\[K_3(\omega) = \frac{\tau_2 \omega}{\sqrt{1 + (\tau_2 \omega)^2}}, \quad \varphi_3(\omega) = \arctan \frac{1}{\tau_2 \omega}.\] (5a)

It is apparent that the resulting AFC and PFC of the oscillatory system can be written in the form

\[K(\omega) = K_1(\omega)K_2(\omega)K_3(\omega), \quad \varphi(\omega) = \varphi_1(\omega) + \varphi_2(\omega) + \varphi_3(\omega).\] (6)

We will show that the circuit determined by Eqs. (6) has the required properties. We do this by examining the graphs of the frequency characteristics in Fig. 1, which were constructed with the following values for the parameters: \( \varepsilon = 0.6, \tau_1 = 0.166(6), \tau_2 = 20 \). Since the excitation band of the system is defined as the difference between the maximum and minimum roots of the equation \( kK(\omega) = 1 \), the AFC shown in the figure ensures the generation of random oscillations at any frequency that satisfies the phase balance condition \( \varphi(\omega) = 0 \) with a suit-