An Algebraic Approach to Discrete Dilations. Application to Discrete Wavelet Transforms

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ABSTRACT. We investigate the connections between continuous and discrete wavelet transforms on the basis of algebraic arguments. The discrete approach is formulated abstractly in terms of the action of a semidirect product $A \times \Gamma$ on $\ell^2(\Gamma)$, with $\Gamma$ a lattice and $A$ an abelian semigroup acting on $\Gamma$. We show that several such actions may be considered, and investigate those which may be written as deformations of the canonical one. The corresponding deformed dilations (the pseudodilations) turn out to be characterized by compatibility relations of a cohomological nature. The connection with multiresolution wavelet analysis is based on families of pseudodilations of a different type.

1. Introduction

Wavelet analysis and related methods have received considerable attention during the past 10 years, both in theoretical and applied sciences. In our opinion, the main reason of such a success is, besides the relative simplicity of the tool, its extreme computational efficiency. The latter is deeply related to the so-called multiresolution structure associated to wavelets, responsible for the existence of fast and accurate algorithms for computing wavelet transforms (the so-called pyramid algorithms). The interested reader may refer to [27, 28] for a detailed discussion of these algorithmic aspects. The main point here is that the wavelet decompositions considered in such an approach are intrinsically discrete decompositions.

Besides such an algorithmic point of view, wavelets have also received great attention in the mathematical physics community [26], especially because of their deep relation with coherent states (see [1] for a recent review). In that approach, the emphasis is put on the symmetries, in particular the fact that the wavelet transform, i.e., the transform which maps a function into the coefficients of its decomposition with respect to a family of wavelets, is covariant with respect to a group action: the affine group of the real line in the case of usual wavelets, or bigger groups in more general situations. In all cases, the groups under consideration are continuous groups. The use of such symmetries is at

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least implicit in many applications, for example in signal analysis (see, for example, the discussions in [7, 22]).

Remarkably enough, little is known about the connections between the two above mentioned approaches (nevertheless see [15] for a formulation of Gabor transform on locally compact abelian groups), at least as far as algebraic arguments are concerned. In particular, the pyramid algorithms, which are obviously attached to rigid algebraic structures, do not seem to have been considered as such (nevertheless see the discussion of embedding of sampling spaces in [18]). The goal of this paper is to start developing such a point of view, and to make a connection with the group theoretical approach. Part of the results presented here are not new from the wavelet point of view, at least in the case of the usual one-dimensional multiresolution wavelet theory. However, we believe that our construction gives a different perspective on the theory, for it makes the connection with different fields (e.g., homological algebra), which may prove useful for concrete problems (for example the complete classification of perfect reconstruction quadrature mirror filters). It also poses the problem of generalizing multiresolution analyses, in terms close to those developed by [4, 11].

Making the connection between the group theoretical approach and the multiresolution approach to wavelets is not an easy task, since no affine group action is available in the discrete case (unless one limits the analysis to discrete fields such as \( \mathbb{Z}/p\mathbb{Z} \), with \( p \) a prime number, as was done in [10, 20]). In this paper, we investigate situations where a semigroup \( \mathcal{A} \) acts on a lattice \( \Gamma \), and we consider the possible actions of the corresponding affine semigroup \( \mathcal{A} \times \Gamma \) on \( \ell^2(\Gamma) \). We show in Section 2 that besides the natural action of translations and the "dilations with holes" (see, e.g., [10, 19, 24]), several such actions may be constructed, based upon pseudodilations \( \mathcal{D}_a, a \in \mathcal{A} \) acting on \( \ell^2(\Gamma) \), generalizing the pseudodilations in [10] to the semigroup context. We introduce two families of pseudodilations (the principal pseudodilations and the associated pseudodilations), and investigate their properties. Such pseudodilations turn out to provide the appropriate setting for developing an abstract version of sub-band coding schemes.

In Section 3 we address the problem of existence of such pseudodilations in a more abstract setting. More precisely, we show that such a problem may be formulated as a problem of representation-valued semigroup cohomology. We develop the corresponding cohomology theory, and show that among the known multiresolution analyses, some may be obtained from trivial 1-cocycles, i.e., 1-coboundaries.

Finally, in Section 4 we address the problem of connection between the continuous and discrete approaches to wavelet analysis. We show that given a pseudodilation, the corresponding sampling of a continuous wavelet transform may be seen as a kind of generalized intertwiner between the action of the continuous affine group on \( L^2(\mathbb{R}) \) and an action of the discrete affine semigroup on \( \ell^2(\mathbb{Z}) \). From our point of view, this approach may be understood as a first step towards a particular aspect of the harmonic analysis of the affine group, namely the decomposition of an irreducible representation of an affine group as a direct sum of representations of a discrete affine subsemigroup.

We notice that several different discrete versions of wavelet analysis have been considered. For the sake of simplicity, we limit the present discussion to the case of the multiresolution wavelet transforms as developed in Daubechies' book [9]. However, we emphasize that a similar approach may be developed in the case of the dyadic wavelet transform, as considered by Mallat [22], very much in the spirit of the Littlewood–Paley decompositions used by harmonic analysts since the early 1930s. Notice also that the formalism developed in this paper does not seem, at the present stage, to yield directly new wavelet bases of sub-band coding schemes. Rather this paper aims at analyzing the algebraic structures associated to multiresolution, so as to pave the way towards generalizations. The latter may, for example, include (separable or non-separable) wavelet bases associated with nontrivial dilations in more than one dimension, or the wavelet bases associated to aperiodic tilings developed for studying quasicrystals in [5, 12], for which the semigroup to be considered is not a semidirect product anymore.