An Operational Calculus for the Euclidean Motion Group with Applications in Robotics and Polymer Science

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ABSTRACT. In this article we develop analytical and computational tools arising from harmonic analysis on the motion group of three-dimensional Euclidean space. We demonstrate these tools in the context of applications in robotics and polymer science. To this end, we review the theory of unitary representations of the motion group of three dimensional Euclidean space. The matrix elements of the irreducible unitary representations are calculated and the Fourier transform of functions on the motion group is defined. New symmetry and operational properties of the Fourier transform are derived. A technique for the solution of convolution equations arising in robotics is presented and the corresponding regularized problem is solved explicitly for particular functions. A partial differential equation from polymer science is shown to be solvable using the operational properties of the Euclidean-group Fourier transform.

1. Introduction

The Euclidean motion group, SE(N),\(^1\) is the semidirect product\(^2\) of \(\mathbb{R}^N\) with the special orthogonal group, \(SO(N)\). That is, \(SE(3) = \mathbb{R}^3 \triangleright SO(3)\). We denote elements of \(SE(N)\) as \(g = (a, A) \in SE(N)\) where \(A \in SO(N)\) and \(a \in \mathbb{R}^N\). The group law is written as \(g_1 \circ g_2 = (a_1 + A_1a_2, A_1A_2)\), and \(g^{-1} = (-A^T a, A^T)\). Alternately, one may represent any element of \(SE(N)\) as an \((N + 1) \times (N + 1)\) homogeneous transformation matrix of the form:

\[
H(g) = \begin{pmatrix}
A & a \\
0^T & 1
\end{pmatrix}.
\]

Clearly, \(H(g_1)H(g_2) = H(g_1 \circ g_2)\) and \(H(g^{-1}) = H^{-1}(g)\), and the mapping \(g \to H(g)\) is an isomorphism between \(SE(N)\) and the set of homogeneous transformation matrices.

Keywords and Phrases. convolution, Euclidean group, rigid body motion, diffusion equations, inverse problems, regularization, Fourier transform, harmonic analysis.

\(^1\)The notation \(SE(N)\) comes from the terminology Special Euclidean group of \(N\) dimensional space.

\(^2\)The notation \(\triangleright\) is used to denote semidirect product, as in [17]. We note that \(\ltimes\) and \(\rtimes\) are more commonly used.
The motion group plays a central role in the kinematic geometry of mechanisms [1, 20, 3], robots [27, 30, 31], and machines [18]. $SE(3)$ and related groups are also important in computer vision and image processing [24, 19, 29]. In the past 40 years, the representation theory and harmonic analysis for the Euclidean group has been developed in the pure mathematics and mathematical physics literature. The study of matrix elements of irreducible unitary representation of $SE(3)$ was initiated by N. Vilenkin [34] in 1957 (some particular matrix elements are also given in [35]). The most complete study of $\tilde{SE}(3)$ (the universal covering group of $SE(3)$) with applications to harmonic analysis was given by W. Miller in [28]. The representations of $SE(3)$ were also studied in [33, 32, 21, 17].

However, despite considerable progress in the representation theory of $SE(3)$ and other non-compact noncommutative groups, these achievements have not yet been widely incorporated in engineering and applied fields. In this article we try to fill this gap. We review the representation theory of $SE(3)$, derive the matrix elements of the irreducible unitary representations and define the Fourier transform for $SE(3)$. We derive new symmetry and operational properties of the Fourier transform and give explicit examples of Fourier transforms of functions on the motion group. We apply noncommutative harmonic analysis to two problems: (1) the solution of equations of the form

$$ (f_1 * f_2)(g) = f_3(g), \quad (1.1) $$

where $*$ denotes the convolution of functions on $SE(3)$, $f_1(g)$ and $f_3(g)$ are known functions, and $f_2(g)$ is to be determined; and (2) solutions to partial differential equations of the form

$$ \frac{\partial f}{\partial L} = Df \quad (1.2) $$

that arise in polymer science where $f(g; L)$ is a probability density function on $SE(3)$ for each value of arclength $L$ and $D$ is an operator explained in Section 2.2.

Techniques for solving (1.1) where $g \in SE(2)$ were presented in [5]. We now consider the more complicated case of when $g \in SE(3)$. Because problem (1.1) is ill-posed, we must seek regularization techniques for performing approximate deconvolution. The approach taken here is to generalize Tikhonov regularization (see e.g., [16]) to the Euclidean group. Our approach to the solution of (1.2) is analogous to the way in which linear diffusion equations with constant coefficients are solved on the line. In order to solve both of these problems, two analytical tools are required: (1) the appropriate concept of Fourier transform; and (2) an understanding of how differential operators acting on functions on the group transform to algebraic operations on the Fourier transform. This is what we refer to as the operational calculus.

The remainder of this article is structured as follows. In Section 2 we illustrate a physical situation in which equations of the form of (1.1) and (1.2) appear. Section 3 reviews the representation theory of $SE(3)$. Sections 4 and 5 define the Fourier transform for $SE(3)$ and derive operational and symmetry properties of the Fourier transform of real-valued functions on $SE(3)$. In Section 6, explicit examples of Fourier transforms on $SE(3)$ are given. In Sections 7 and 8, operational properties derived in this article are used to regularize the solution of (1.1), and provide analytical solutions to (1.2), respectively.

2. Motivational Examples

Here we present in greater detail problems that motivate the need for an operational calculus for the Euclidean motion group.