EXACT SOLUTION OF PROBLEMS OF FLOW THEORY WITH ISOTROPIC-KINEMATIC HARDENING.
PART 1. SETTING THE LOADING TRAJECTORY IN THE SPACE OF STRESSES

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For an arbitrary isotropic and linear kinematic hardening and loading paths given in the form of arbitrary multisection polygonal lines in the five-dimensional deviator space of stresses, we studied analytically an initially isotropic elastoplastic von Mises material and the associated flow rule. The solutions obtained are valid for arbitrary relationships governing the variation of the spherical component of the stress tensor. Explicit solutions are obtained for several important cases of material behavior.

As is well known, plastic behavior of a number of materials can be adequately described only by the combined (isotropic-kinematic) rule of hardening rather than by models presuming either strictly isotropic or strictly kinematic hardening behavior. Models with combined hardening are widely used for numeric calculations as well [1–3]. Theoretical study and practical application of materials with isotropic-kinematic hardening would be simplified significantly if the plastic behavior of these materials could be described analytically. However, due to the mathematical complexity of this problem, analytical studies were carried out only separately for isotropic and kinematic laws of hardening, and explicit solutions were obtained mainly for the isotropic case [4–9]. As for the kinematic hardening law, only few analytical results are available at present [10–12].

Among other loading paths, of special interest are multisection polygonal lines, since they are widely used in the experiments where investigation of plastic behavior is carried out both in the spaces of total [6, 13] and plastic strains [14], and in the space of stresses [15, 16]. Studies of cyclic loading histories [17–21] show that deformation trajectories in these cases are multisection polygonal lines as well.

The present paper is dedicated to analytical study of an initially elastoplastic material which is described by the theory of flow combining the von Mises yield condition, the associated flow rule, and isotropic-kinematic strain hardening. We obtained solutions for loading paths represented by multisection polygonal lines in the deviatoric space of stresses, provided that the variation of the stress tensor spherical component is arbitrary, isotropic hardening is arbitrary as well, while kinematic hardening is a linear function of plastic strain.

**Basic Set of Equations.** The analytical study is based on the following set of equations of the theory of flow with isotropic-kinematic strain hardening [22]:

\[ e_{kj} = e_{kj}^e + e_{kj}^p, \]
\[ \varepsilon = \sigma / K, \quad e_{kj}^e = S_{kj} / 2G, \]
\[ f_p = \bar{\sigma}_p - \Phi(q) = 0, \]
\[ de_{kj}^p = \frac{d\lambda}{2G} (s_{kj} - \rho_{kj}). \]
Here \( e_{kj} \) are the components of the total strain deviator, which according to [1] can be represented by the sum of elastic \( e_{kj}^e \) and plastic \( e_{kj}^p \) strain components, and Hooke’s law (2) holds for deviators \( e_{kj}^e \) and \( s_{kj} \), as well as for the spherical components of elastic strain, \( \varepsilon \), and stress, \( \sigma \), tensors, where \( G \) and \( K \) are, respectively, the shear and bulk moduli. In the von Mises yield condition (3), the expression

\[
\bar{\sigma}_p = \left[ \frac{3}{2} (s_{kj} - p_{kj})^2 \right]^{1/2}
\]

(6)
denotes the equivalent active stress, and \( \Phi(q) > 0 \) is a certain preset function of isotropic strain hardening which depends on the Odquist parameter \( q \):

\[
q(x) = \int_0^x \dot{e}^p \, dx', \quad dq = de^p,
\]

(7)

where \( x \) is a measure of length of the loading path in the space of stresses or strains, and \( de^p \) is the equivalent increment of the plastic strain:

\[
\bar{de}^p = \left( \frac{2}{3} e_{kj}^p e_{kj}^p \right)^{1/2}
\]

(8)

\( \dot{e}^p \) is the equivalent “rate” of the plastic strain:

\[
\bar{\dot{e}}^p = \left( \frac{2}{3} e_{kj}^p e_{kj}^p \right)^{1/2}
\]

(9)

(hereinafter the overdot denotes the operation of differentiation with respect to \( x \)).

The set of equations also includes the flow rule (4) associated with (3), where \( \lambda \) is the dimensionless Lagrange multiplier, \( de_{kj}^p \) is the tensor of the plastic-strain increment, and \( p_{kj} \) are current coordinates of the locus of the von Mises yield surface, which moves in the process of plastic straining. The applied relationship (5) describes a linear kinematic strain hardening characterized by a preset coefficient \( \eta \geq 0 \), which is given in the dimensionless form in order to simplify further calculations.

Hereinafter, subscripts \( kj \) denote tensor components in a certain coordinate system set by three orthogonal coordinate axes \( (k, j = \{1, 2, 3\}) \), and a standard convention is adopted concerning summation by the recurring indices.

As is known, for any point of an arbitrary loading path

\[
\bar{\sigma}_p = \Phi(q), \quad \frac{\partial f_p}{\partial s_{kj}} ds_{kj} > 0 \quad \text{(active process)},
\]

(10)

\[
\bar{\sigma}_p = \Phi(q), \quad \frac{\partial f_p}{\partial s_{kj}} ds_{kj} < 0 \quad \text{(unloading)},
\]

(11)

\[
\bar{\sigma}_p = \Phi(q), \quad \frac{\partial f_p}{\partial s_{kj}} ds_{kj} = 0 \quad \text{(neutral process)},
\]

(12)

\[
\bar{\sigma}_p < \Phi(q) \quad \text{(purely elastic process)},
\]

(13)