ANALYTIC CORRECTION OF AN INADMISSIBLE STRESSED STATE IN THE NUMERICAL ANALYSIS OF ELASTOPLASTIC DEFORMATION OF RIGID BODIES

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We propose closed analytic expressions reducing the stresses typical of a physically inadmissible state obtained as a result of the numerical solution of a quasistatic elastoplastic problem to the state corresponding to the yield surface. The Lagrange functional is modified to remove the excess stresses at the points where the condition of yield is violated, which enables one to maintain the equilibrium state of the elastoplastic body in each stage of loading.

In the process of numerical solution of elastoplastic problems, it is sometimes necessary to correct a current inadmissible stressed state \( g(\sigma_y) > 0 \) in order to transform it into the state corresponding to the yield surface \( g(\sigma_y) = 0 \). It is known that this may happen even for quite small levels of loading and can be explained both by the discrete character of loads and linearization of the determining relations and by the accumulation of computational errors.

At present, the operation of correction of stresses in computational schemes is, as a rule, based on check tests followed (if necessary) by the required correction of stresses in each loading stage. It includes the procedure of determining the coefficients used as multipliers of the current inadmissible stressed state [1, 2]. Moreover, in the case where the conditions of simple loading are not satisfied due to the physical nonlinearity of the problem, these coefficients undergo additional correction [3, 4]. It is well known that this simple, at first sight, operation not only complicates the algorithm of numerical computation of elastoplastic problems but also requires significant amounts of computer time, especially in analyzing multi-dimensional discrete computational schemes.

The aim of the present work is to deduce a closed finite set of relations reducing (in our case, projecting) stresses onto the yield surface. To do this, we use only analytic methods and, if necessary, some generally accepted hypotheses and conjectures from the theory of elastoplastic media. Our presentation is based on the use of the associated law of flow for isotropic materials with hardening and the Huber-von Mises yield surface.

1. Mathematical Formulation of the Elastoplastic Problem. Consider an arbitrary multiply connected region \( V \) related to a Cartesian coordinate system \( OXYZ \) and bounded by a surface \( \Gamma \). On one part of this surface \( \Gamma_u \), we specify a vector of displacements and, on the remaining part of the surface \( \Gamma_\sigma \), we specify a vector of surface loads. Thus, we have \( \Gamma = \Gamma_u \cup \Gamma_\sigma \). In the case of small elastoplastic strains, at each point \( x \) of the body with coordinates \( x_i \), we have the differential equilibrium equations

\[
\sigma_{ij,j} + K_i, \quad x \in V, \quad i, j = 1, 2, 3, \tag{1}
\]

supplemented by the following relationships between the level of strains and displacements:

\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad x \in V. \tag{2}
\]

The static boundary conditions on the surface \( \Gamma_\sigma \) and the kinematic boundary conditions on the surface \( \Gamma_u \) have the form

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\[ \sigma_j n_j = P_i, \quad x \in \Gamma_\sigma, \]
\[ u_i(x_i) = u_i^0, \quad x \in \Gamma_u, \]

where \( K_i \) are the components of the vector of bulk forces, \( P_i \) are the components of the vector of loading defined on \( \Gamma_\sigma \), \( n_j \) are the direction cosines of the outer normal to the boundary of the region \( V \), and \( u_i = u_i(x) \) are displacements of points of the body.

We describe the elastoplastic behavior of the material within the framework of the associated rule of flow [5, 6]:

\[ \delta e = A \delta \sigma + d \lambda \nabla \sigma. \]  

Relations (5) connect the increments of the components of the vector of small elastoplastic strains \( \delta e \) with the components of the vector of stresses \( \sigma \) and the vector of increments of stresses \( \delta \sigma \). Here, \( A \) is the matrix of elasticity coefficients, \( \nabla \) is the operator of gradient of the yield function \( g(\sigma) \) in the nine-dimensional space of stresses, i.e., \( \nabla \sigma = \left( \frac{\partial g}{\partial \sigma_{ij}} \right) \).

The parameter \( d \lambda \) corresponding to the plastic part of strains \( \delta e^p = d \lambda \nabla \sigma \) differs from zero in the plastic region and can be determined from the condition \( (d \sigma, \delta e^p) = 0 \) as follows:

\[ d \lambda = (\delta e, B \nabla \sigma) / [(\nabla \sigma, B \nabla \sigma) + H], \quad B = A^{-1}, \]

where \( H \) is the slope of the stress-strain diagram of the material for a current stress-strain state. It is computed as the scalar product

\[ H = \left( \begin{array}{c} \frac{\partial g}{\partial \epsilon_{ij}} \\ \frac{\partial g}{\partial \epsilon_{ij}} \\ \frac{\partial g}{\partial \sigma} \\ \frac{\partial g}{\partial \sigma} \\ \end{array} \right) = (\nabla \sigma, \nabla \sigma). \]  

For the sake of simplicity, we denote \( \frac{\partial g}{\partial \epsilon_{ij}} = \nabla \sigma_{ij} \). Then

\[ H = \left( \begin{array}{c} \frac{\partial g}{\partial \epsilon_{ij}} \\ \frac{\partial g}{\partial \epsilon_{ij}} \\ \frac{\partial g}{\partial \sigma} \\ \frac{\partial g}{\partial \sigma} \\ \end{array} \right) = (\nabla \sigma_{ij}, \nabla \sigma). \]

We also introduce a vector \( f = \{f_i\}, i = 1, 2, \ldots, 6 \), expressed via the vector of normal to the yield surface \( \nabla \sigma \) and the matrix \( B \) by the formula

\[ f = \begin{cases} B \nabla \sigma / \sqrt{(\nabla \sigma, B \nabla \sigma) + H} & \text{for } \begin{cases} g = 1, \quad d g = 0, \\
                           g < 1 \text{ or } \quad g = 0, \quad d g < 0. \end{cases} \\
0 & \end{cases} \]

Note that, for the perfect elastoplastic material, the value of \( H \) is equal to zero, whereas for the elastic material, it tends to infinity. Thus, the determining equations (5) resolved with respect to the increment of stresses \( \delta \sigma \) can be written in the form [7, 8]

\[ d \sigma = D \delta e, \quad D = B + C, \quad C = \{f_i f_j\}, \quad i, j = 1, 2, \ldots, 6, \]

where \( D \) is a matrix that describes the elastoplastic behavior of the material. In our case, for points from the elastic part of the region, as well as for the case of neutral loading and unloading, we have \( f = 0 \).

Assume that the numerical solution of the quasistatic elastoplastic problem posed above is realized by the method of finite elements based of the Lagrange variational principle: