Some Essential Properties of $Q_p(\partial \Delta)$-Spaces

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Abstract. For $p \in (-\infty, \infty)$, let $Q_p(\partial \Delta)$ be the space of all complex-valued functions $f$ on the unit circle $\partial \Delta$ satisfying

$$\sup_{I \subset \partial \Delta} |I|^{1-p} \int_I \int_I \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}} |dz||dw| < \infty,$$

where the supremum is taken over all subarcs $I \subset \partial \Delta$ with the arclength $|I|$. In this paper, we consider some essential properties of $Q_p(\partial \Delta)$. We first show that if $p > 1$, then $Q_p(\partial \Delta) = \text{BMO}(\partial \Delta)$, the space of complex-valued functions with bounded mean oscillation on $\partial \Delta$. Second, we prove that a function belongs to $Q_p(\partial \Delta)$ if and only if it is Mobius bounded in the Sobolev space $L^2_p(\partial \Delta)$. Finally, a characterization of $Q_p(\partial \Delta)$ is given via wavelets.

1. Introduction

Throughout this paper, suppose that $\Delta$, $\tilde{\Delta}$, and $\partial \Delta$ are the open unit disk, the closed unit disk, and the unit circle in the finite complex plane $\mathbb{C}$. For $p \in (-\infty, \infty)$, let $Q_p(\partial \Delta)$ be the space of all Lebesgue measurable functions $f : \partial \Delta \to \mathbb{C}$ with

$$\|f\|_{Q_p(\partial \Delta)} = \sup_{I \subset \partial \Delta} |I|^{-p} \int_I \int_I \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}} |dz||dw|^{1/2} < \infty,$$

where the supremum is taken over all subarcs $I \subset \partial \Delta$ of the arclength $|I|$. Note that if $p = 2$, then $Q_p(\partial \Delta) = \text{BMO}(\partial \Delta)$, John–Nirenberg's space of functions having bounded mean oscillation on $\partial \Delta$. A Lebesgue measurable function $f : \partial \Delta \to \mathbb{C}$ is in $\text{BMO}(\partial \Delta)$ [8] if and only if

$$\|f\|_{\text{BMO}(\partial \Delta)} \equiv \sup_{I \subset \partial \Delta} |I|^{-1} \int_I |f(z) - f_I|^2 |dz|^{1/2} < \infty,$$ 

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where the supremum ranges over all subarcs \( I \subset \partial \Delta \) and \( f_I \) stand for the average of \( f \) over \( I \)

\[
f_I = \frac{1}{|I|} \int_I f(z) |dz|. \]

Recall that the space \( Q_p(\partial \Delta) \), \( p \in (0, 1) \) was introduced in [5] (there it was written as \( Q'_p \)) when Essén and Xiao studied the boundary behavior of the holomorphic \( Q_p \)-space [1], which is the set of all holomorphic functions \( f \) on \( \Delta \) obeying

\[
\|f\|_{Q_p} = \sup_{w \in \Delta} \left[ \int_{\Delta} |f'(z)|^p \left[ 1 - \left| \phi_w(z) \right|^2 \right]^{\frac{p}{2}} dxdy \right]^{\frac{1}{p}} < \infty, \quad z = x + iy. \tag{1.3} \]

Here and henceforth,

\[
\phi_w(z) = \frac{w - z}{1 - \overline{w}z} \tag{1.4} \]

is a Möbius transform sending \( w \) to 0, and \( dxdy (z = x + iy) \) means the two-dimensional Lebesgue measure on \( \Delta \). Later on, Poisson extension to \( \Delta \), \( \bar{\partial} \)-equations, and a Fefferman–Stein type decomposition of \( Q_p(\partial \Delta) \), \( p \in (0, 1) \) were established by Nicolau and Xiao in [11]. As a continuation of [5], Janson discussed the dyadic analog of \( Q_p(\partial \Delta) \), \( p \in (0, 1) \) [7].

The major purpose of the present paper is to investigate some essential properties of \( Q_p(\partial \Delta) \). First, in Section 2 we show that \( Q_p(\partial \Delta) \) is nondecreasing with \( p \), in particular \( Q_1(\partial \Delta) = BMO(\partial \Delta) \) or \( C \) when \( p > 1 \) or \( p < -1 \). Next, in Section 3 we reveal that \( Q_p(\partial \Delta) \) is a Möbius bounded subspace of the Sobolev space on \( \partial \Delta \). Finally, we give a description of \( Q_p(\partial \Delta) \) in terms of wavelets.

Throughout this paper, the letters \( C \) and \( c \) denote different positive constants which are not necessarily the same from line to line. Moreover, \( A \approx B \) means that there are two constants \( C \) and \( c \) independent of both \( A \) and \( B \) to ensure \( cA \leq B \leq CA \). Also, for an \( r \in (0, \infty) \) and a subarc \( I \), \( rI \) represents the subarc with the same center as \( I \) and with the length \( r|I| \).

## 2. Monotonicity

In this section, we focus on the monotonicity of \( Q_p(\partial \Delta) \) and discover that the case \( p \in (0, 1] \) is of independent interest.

### Theorem 1.

Let \( p \in (-\infty, \infty) \). Then \( Q_p(\partial \Delta) \) is nondecreasing with \( p \). In particular,

(i) If \( p \in (-\infty, -1] \), then \( Q_p(\partial \Delta) = C \).

(ii) If \(-1 < p_1 \neq p_2 \leq 1 \), then \( Q_{p_1}(\partial \Delta) \neq Q_{p_2}(\partial \Delta) \) and \( Q_1(\partial \Delta) \neq BMO(\partial \Delta) \).

(iii) If \( p \in (1, \infty) \), then \( Q_p(\partial \Delta) = BMO(\partial \Delta) \).

**Proof.** Let \( p_1 < p_2 \). If \( f \in Q_{p_1}(\partial \Delta) \), then for any subarc \( I \subset \partial \Delta \),

\[
\int_I \int_I \frac{|f(z) - f(w)|^2}{|z - w|^{2-p_2}} |dz| |dw| \leq |I|^{p_2-p_1} \int_I \int_I \frac{|f(z) - f(w)|^2}{|z - w|^{2-p_1}} |dz| |dw| \leq |I|^{p_2} \|f\|^2_{Q_{p_1}(\partial \Delta)},
\]

namely, \( f \in Q_{p_2}(\partial \Delta) \). So, \( Q_{p_1}(\partial \Delta) \subset Q_{p_2}(\partial \Delta) \).