STABILITY OF ZAKIAN $I_{MN}$ RECURSIONS FOR LINEAR DELAY DIFFERENTIAL EQUATIONS *

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Abstract.

Long sequences of linear delay differential equations (DDEs) frequently occur in the design of control systems with delays using iterative-numerical methods, such as the method of inequalities. Zakian $I_{MN}$ recursions for DDEs are suitable for solving this class of problems, since they are reliable and provide results to the desired accuracy, economically even if the systems are stiff. This paper investigates the numerical stability property of the $I_{MN}$ recursions with respect to Barwell's concept of $P$-stability. The result shows that the recursions using full grade $I_{MN}$ approximants are $P$-stable if, and only if, $N - 2 \leq M \leq N - 1$.

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1 Introduction.

Long sequences of linear functional equations occur frequently in the design of control systems by iterative-numerical methods, such as the method of inequalities [44, 43] (see also [21, 23, 32]). Because such a sequence may contain thousands of elements that can be stiff, a numerical method for solving the equations is required not only to be reliable in the sense that it preserves the asymptotic stability properties of the systems, but also to be economical, especially for large systems.

For the design of control systems involving time-delays, their mathematical models are described by linear delay differential equations (DDEs) with constant coefficients and fixed delays (see, for example, [27, 22]). It is found in practice that Zakian $I_{MN}$ recursions for DDEs are reliable and provide results to the desired accuracy, economically even if the systems are stiff. Algorithms based on the $I_{MN}$ recursions for DDEs are used satisfactorily in the design of control systems with delays (see [3]). Apparently, there are known reliable methods which are applicable to a wider class of DDEs (e.g. nonlinear, time-varying delays) (see, for example, [50, 15]). But for long sequences of linear DDEs, the

$I_{MN}$ recursions are preferable. Detailed reasons for this can be found in [2] (see also [3]) and relate to economy of computation (see the discussions in Sections 2.2 and 2.3).

The purpose of this paper is to establish the numerical stability property of the $I_{MN}$ recursions for DDEs, with respect to Barwell’s [6] concept of $P$-stability, so as to ensure that the recursions are reliable in practice. The key tool in the stability analysis is a theorem due to Zennaro [50]. It is shown that the $I_{MN}$ recursions for DDEs are $P$-stable if, and only if, $M$ is chosen such that $N - 2 \leq M \leq N - 1$ and $M \geq 0$.

The organization of the paper is as follows. To provide sufficient background, a review of Zakian $I_{MN}$ approximants and recursions is presented in Section 2; this includes the $I_{MN}$ recursions for linear DDEs. In Section 3, the numerical stability analysis of the $I_{MN}$ recursions for DDEs is given and the main result is stated in Theorem 3.3.

2 Zakian $I_{MN}$ approximants and recursions.

2.1 Definition and properties of $I_{MN}$.

Let $\mathcal{X}$ denote the linear space of functions $x$ such that $x$ is continuous on $[0, \infty)$ and, for some real $\sigma$, $x(t) = \mathcal{O}(\exp(\sigma t))$, $t \to \infty$. Zakian [34, 38] defines the $I_{MN}$ approximant of $x$ evaluated at $t$ by the improper integral

$$I_{MN}(x, t) \triangleq \int_0^\infty x(\lambda t) \sum_{i=1}^N K_i \exp(-\alpha_i \lambda) \, d\lambda, \quad t \in T,$$

where ($\alpha_i, K_i$) are defined constants, $T$ is the set of all $t \in [0, \infty)$ such that the integral exists and the nonnegative integers ($M, N$) are, respectively, the orders of the numerator and denominator of the rational function which is the Laplace transform of $\sum_{i=1}^N K_i \exp(-\alpha_i \lambda)$. Ideally, it is required that $I_{MN}(x, t) = x(t)$ for all $t \geq 0$ and for all $x \in \mathcal{X}$. Since this is not possible, it is instead required that the constants ($\alpha_i, K_i$) be chosen so that the operator $I_{MN}$ is, in some sense, the best approximant to the identity operator. Accordingly, the constants ($\alpha_i, K_i$) which depend on $M$ and $N$ can be defined in various ways by using different criteria of best approximation.

In this work, attention is restricted only to full grade $I_{MN}$ approximants [34, 38], whose constants ($\alpha_i, K_i$) are defined by the following rules:

$$\sum_{i=1}^N K_i \frac{z^{N-i}}{(z + \alpha_i)^N} = e_{MN}^z,$$

$$\alpha_{MN} \triangleq \min_i \{\Re(\alpha_i)\} > 0,$$

where $e_{MN}^z$ is the $(M/N)$ Padé approximant to $\exp(-z)$, given explicitly by

$$e_{MN}^z = \frac{B_{NM}(-z)}{B_{MN}(z)}, \quad B_{MN}(z) = \sum_{k=0}^N \frac{(M + N - k)!}{(M + N)!} \frac{N!}{(N - k)!} \frac{z^k}{k!}.$$