We use energy methods to prove the existence and uniqueness of solutions of the Dirichlet problem for an elliptic nonlinear second-order equation of divergence form with a superlinear term \[ i.e., \quad g(x, u) = v(x) a(x) u^{p-1} u, \quad p > 1 \] in unbounded domains. Degeneracy in the ellipticity condition is allowed. Coefficients \( a_{i,j}(x, r) \) may be discontinuous with respect to the variable \( r \).

1. Introduction

This work is a continuation of [1]. Let \( \Omega \) be an open subset of \( \mathbb{R}^m (m \geq 2) \). We consider a strongly nonlinear degenerate elliptic equation of the type

\[
- \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{m} a_{i,j}(x,u) \frac{\partial u}{\partial x_j} \right) + a(x)(v(x)|u|^{p-1}u) = f, \tag{*}
\]

where \( p \) is a real number greater than 1. The equation is degenerate elliptic if a condition of the form

\[
\sum_{i,j=1}^{m} a_{i,j}(x,r) \xi_i \xi_j \geq v(x)|\xi|^2
\]

is satisfied for every \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m \), a.e. \( (x, r) \in \Omega \times \mathbb{R} \). When \( \Omega \) is bounded, it is known that there exists a unique weak bounded solution for the Dirichlet problem related to (*) (see [1]). In this paper, we extend these results to an unbounded set \( \Omega \) and on other data are minimal. Our results are in some respects similar to those of Guglielmino and Nicolosi [2] and generalize the ones obtained by Ivanov and Mkrtycjan [3]. The main difference of our results comes from the fact that we merely use energy methods which allow us to treat a greater class of functions \( f \). So we do not need, as in [2] and [3], any hypothesis on the growth of the data at infinity. Moreover, we replace the continuity hypothesis of coefficients \( a_{i,j}(x, r) \) with respect to the variable \( r \) with a weaker one (see [4]). In the nondegenerate case, \( v(x) = 1 \), the first result where the existence and uniqueness of a solution of the problem (*) was given without growth at infinity on \( f \) is due to Brezis [5]; then, this result was generalized in [6] and [7]. Our argument has some points of contact with the one introduced by Díaz and Oleinik in [7]. More precisely, for every \( R > 0 \), we define

\[
B_R = \{ x \in \mathbb{R}^m : |x| < R \},
\]

\[
\Omega_R = \Omega \cap B_R,
\]

and we consider the unique solution \( u_N \) of problem (*) in \( \Omega_N \). Then, taking \( u_N \Lambda_R \) as a test function in the
integral equality satisfied by \( u_N \) \((2R < N)\), we obtain an \textit{a priori} estimate from above for the norm of \( u_N \) in \( H^1_\nu(\Omega_R) \cap L^{p+1}(v(x), \Omega_R) \) (Lemma (5.1)); here

\[
\Lambda_R = \theta^2 \left( \frac{|x|}{R} \right),
\]

where \( \theta \in C^\infty(\mathbb{R}) \) is a cutoff function such that \( \theta'(s) = O\left((1 - s)^{t-1}\right) \) \((t > 0)\). Finally, by diagonal extraction we obtain a weak solution of (*) with Dirichlet data. The uniqueness of the solution is obtained in the same manner of [7], assuming that \( a_{i,j} \) does not depend on \( r \).

2. Function Spaces

Let \( \mathbb{R}^m \) be the Euclidean \( m \)-space with generic point \( x = (x_1, x_2, \ldots, x_m) \).

**Hypothesis 2.1.** Let \( v(x) \) be a positive function defined on \( \Omega \); there exists a real number \( g > m/2 \) such that

\[
v(x) \in L^s(\Omega_R), \quad \frac{1}{v(x)} \leq L^g(\Omega_R)
\]

for every \( R > 0 \); here,

\[
s = \frac{mg}{2g - m}.
\]

Let \( D \) be a bounded open subset of \( \mathbb{R}^m \). Denote by \( H^1_\nu(D) \) the completion of \( C^1(\overline{D}) \) with respect to the norm

\[
\| u \|_{1,\nu,D} = \left( \int_D v(x) \{ |u|^2 + |\nabla u|^2 \} \, dx \right)^{1/2}.
\]

\( H^{1,0}_\nu(D) \) is the closure of \( C^\infty_0(D) \) in \( H^1_\nu(D) \). According to Hypothesis 2.1, we get the imbedding

\[
H^{1,0}_\nu(D) \hookrightarrow L^{\tilde{2}}(D)
\]

where

\[
\tilde{2} = \frac{2mg}{mg + m - 2g}
\]

is greater than 2; moreover, the inequality

\[
\left( \int_D v(x) |u|^2 \, dx \right)^{1/2} \leq c(m, g, v) \left( \int_D v(x) |\nabla u|^2 \, dx \right)^{1/2}
\]