ONE CLASS OF SINGULAR COMPLEX-VALUED RANDOM VARIABLES OF THE JESSEN–WINTNER TYPE

O. V. Shkol'nyi and M. V. Prats'ovytyi

We study the structure of the distribution of a complex-valued random variable $\xi = \sum a_k \xi_k$, where $\xi_k$ are independent complex-valued random variables with discrete distribution and $a_k$ are terms of an absolutely convergent series. We establish a criterion of discreteness and sufficient conditions for singularity of the distribution of $\xi$ and investigate the fractal properties of the spectrum.

1. Consider a complex-valued random variable

$$\xi = \sum_{k=1}^{\infty} a_k \xi_k,$$

where $\xi_k$ are independent discrete distributed complex-valued random variables that take values in a bounded at most countable set $E = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_i, \ldots\}$ of complex numbers, i.e., $|\epsilon_i| < a < \infty$, with probabilities $p_{0k}$, $p_{1k}, \ldots, p_{ik}$, respectively, $\sum p_{ik} = 1; p_{ik} > 0$, $a_k \in \mathbb{C}$, and $k = 1, 2, \ldots$, and $a_k$ are terms of an absolutely convergent series.

Random variables of the form (1) for real-valued $\xi_k$ are called random variables of the Jessen–Wintner type. They have a pure distribution (the Jessen–Wintner theorem). The structure and fractal properties of distributions of such random variables were investigated by many authors (see, e.g., [1–5]). In the present paper, we consider similar problems. As far as we know, in the complex-valued case, theorems of the general type (of the type of the Jessen–Wintner and Levy [1] theorems) are not proved.

**Definition 1** (of the types of distribution of complex-valued random variables). A distribution of a complex-valued random variable $\xi$ is called

(i) discrete if the corresponding probability measure $\xi$ of the random variable $P(\cdot)$ is concentrated in an at most countable set;

(ii) continuous if $P(\cdot)$ is defined and equal to zero in every one-point set, in particular, it is singular if there exists a Borel set $A$ such that $\lambda(A) = 0$ and $P(\xi \in A) = 1$, where $\lambda(\cdot)$ is the Lebesgue measure in $R^2$, and the distribution is absolutely continuous if $P(\epsilon \in A) = 0$ for every Borel set $A$ such that $\lambda(A) = 0$.

Also recall that, according to the Lebesgue theorem, every countably additive probability measure $\mu$ is uniquely representable in the form

$$\mu(\cdot) = \alpha_1 \mu_d(\cdot) + \alpha_2 \mu_{ac}(\cdot) + \alpha_3 \mu_s(\cdot),$$

where $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\mu_d$ is a discrete probability measure, $\mu_{ac}$ is an absolutely continuous probability measure, and $\mu_s$ is a probability measure singular with respect to the Lebesgue measure.
Expression (2) is called the structure of the measure $\mu$ (respectively, the structure of the distribution of the random variable associated with $\mu$). If there is $\alpha_i$ equal to one, then the measure (distribution) is called pure (purely discrete, purely absolutely continuous, and purely singular).

The triple $(E, \{a_k\}, \|P_{ik}\|)$ uniquely defines a random variable and the structure of its distribution. We study this structure under the condition that the set $E$ and sequences $\{a_k\}$ are fixed. Our aim is to find conditions that should be imposed on the matrix of probabilities $\|P_{ik}\|$ in order that the distribution of $\xi$ be purely discrete, continuous, and, in particular, purely singular.

2. A number $z$ is a possible value of a random variable $\xi$ if there exists a sequence $\{z_k\}$, $z_k \in E$, such that

$$z = \sum_{k=1}^{\infty} a_k z_k = \sum_{k=1}^{\infty} a_k e_{i_k}.$$  \hspace{1cm} (3)

Note that, in the dependence on $E$ and $\{a_k\}$, it is possible that different sequences $\{z_k\}$ and $\{z_k'\}$ lead to the same value, i.e., $\sum a_k z_k = \sum a_k z_k'$. We give only one trivial example for the real-valued case:

$$\frac{1}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \ldots = \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots.$$  \hspace{1cm} (4)

Furthermore, it is possible to give examples of $E$ and $\{a_k\}$ for which the same $z$ has a continuum set of “different” representations. For example, in the equality

$$z = \left(\frac{z_1}{2} + \frac{z_2}{2}\right) + \left(\frac{z_3}{2^2} + \frac{z_4}{2^2}\right) + \ldots + \left(\frac{z_{2k-1}}{2^k} + \frac{z_{2k}}{2^k}\right) + \ldots,$$

the value $z$ does not change if we replace the value $(0, 1 + i)$ of the pair $(z_{2k-1}, z_{2k})$ by $(1 + i, 0)$.

Definition 2 (N-property). We say that a pair $(E, \{a_k\})$ possesses the $N$-property if, for any complex number $z$, there exists an at most countable set of sequences $\{z_k\}$, $z_k \in E$, such that $z = \sum a_k z_k$.

Lemma 1. If

$$M = \prod_{k=1}^{\infty} \max_i \left\{p_{ik}\right\} > 0,$$  \hspace{1cm} (4)

then the distribution of the random variable $\xi$ is purely discrete.

Proof. Let $\max_i \left\{p_{ik}\right\} = p_{i^*_{(k)}}$. Then the point $z_0 = \sum a_k e_{i^*_{(k)}}$ is an atom of the distribution $\xi$ because, taking into account the remark on the possibility of different representations of numbers in the form (3), we have

$$P\{\xi = z_0\} \geq P\{e_{i^*_{(1)}} = e_{i^*_{(2)}} = e_{i^*_{(3)}} = \ldots\} = \prod_{k=1}^{\infty} p_{i^*_{(k)}} = M > 0.$$