ESTIMATES OF THE KOLMOGOROV WIDTHS FOR CLASSES OF INFINITELY DIFFERENTIABLE PERIODIC FUNCTIONS

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Lower estimates of the Kolmogorov widths are obtained for certain classes of infinitely differentiable periodic functions in the metrics of $C$ and $L$. For many important cases, these estimates coincide with the values of the best approximations of convolution classes by trigonometric polynomials calculated by Nagy, and, hence, they are exact.

In this work, we find lower bounds of the Kolmogorov $N$-widths, i.e., of the quantities

$$d_N(\mathfrak{R}, X) = \inf_{L_N \subset X} \sup_{\xi \in \mathfrak{R}} \inf_{\xi \in L_N} \|\xi - \xi\|_X,$$

where $X$ denotes either the space $L$ of $2\pi$-periodic summable functions $f(\cdot)$ with the finite norm

$$\|f\|_1 = \int_0^{2\pi} |f(t)| dt,$$

or the space $C$ of $2\pi$-periodic continuous functions $f(\cdot)$ with the finite norm $\|f\|_C = \max \{ f(t) \}$, $\mathfrak{R}$ are the classes of functions defined by generalized $(\psi, \beta)$-derivatives introduced by Stepanets (see, e.g., [2]), and $L_N$ denote all possible subspaces of $X$.

**Definition 1** [2, p. 25]. Let $f \in L$ and let

$$s[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

be its Fourier series. Further, let $\psi(k)$ be an arbitrary function of natural argument and let $\beta$ be a fixed real number. If the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left[ a_k \cos \left( kx + \frac{\beta \pi}{2} \right) + b_k \sin \left( kx + \frac{\beta \pi}{2} \right) \right]$$

is the Fourier series of a certain summable function, then this function is called the $(\psi, \beta)$-derivative of the function $f(\cdot)$ and is denoted by $f_\psi^{(\beta)}(\cdot)$, and the set of functions $f(\cdot)$ satisfying this condition is denoted by $L_\beta^\psi$.

Denote

$$L_{\beta, p}^\psi \overset{df}{=} \left\{ f \in L_\beta^\psi : \|f_\psi^{(\beta)}\|_p \leq 1 \right\}, \quad p = 1, \infty, \quad C_{\beta, p}^\psi = C \cap L_{\beta, p}^\psi.$$
We are interested in the case where the function $\psi(k)$ that determines the classes $L^q_{p, r}$ and $C^q_{p, r}$ can be represented in the form

$$\psi(k) = \varphi(k) e^{-\alpha k^r}, \quad 0 < r < 1, \quad \alpha > 0,$$

(1)

where $\varphi(k)$ is a positive nonincreasing function of natural argument.

It follows from Sec. 1.8.2 in [2] that, in this case, the classes $C^q_{p, r}$ consist of infinitely differentiable (but not necessarily analytic) functions. Moreover, it follows from Proposition 1.7.2 in [2] that, in the case under consideration, elements of the class $L^q_{p, r}$ for arbitrary $\beta \in \mathbb{R}$ can be represented almost everywhere as follows:

$$f(x) = \frac{a_0}{2} + \left( \Psi_\beta \ast \varphi \right)(x) = \frac{a_0}{2} + \frac{1}{2\pi} \int_0^{2\pi} \Psi_\beta(x - t) \varphi(t) dt,$$

where

$$\varphi \in L, \quad \int_0^{2\pi} \varphi(t) dt = 0, \quad \| \varphi \|_p \leq 1,$$

and $\Psi_\beta(t)$ is a summable function whose Fourier series has the form

$$s[\Psi_\beta(t)] = \sum_{k=1}^{\infty} \psi(k) \cos \left( \frac{kt - \beta \pi}{2} \right).$$

Order estimates of the widths $d_n(C^q_{p, \omega}, C)$ and $d_n(L^q_{p, 1}, L)$ for $\psi(k)$ of the form (1) were found by Kushpel' (see, e.g., [3, Theorem 5.2, p. 41]).

Note that the problem of determination of exact lower bounds of the widths of classes $C^q_{p, \omega}$ and $L^q_{p, 1}$ in the metrics of $C$ and $L$ in the case where $\psi(k)$ has the form (1) is not adequately studied (the special case $\psi(k) = \varphi(k) e^{-\alpha k}$, i.e., $r = 1$, was considered in [4-7]).

The principal result of this paper is the following statement:

**Theorem 1.** Let $\psi(k)$ have the form (1) and be thrice monotone for $\beta \neq 2p - 1, \quad p \in \mathbb{Z}$, i.e., assume that it satisfies the inequalities

$$\Delta \psi(k) \overset{df}{=} \psi(k) - \psi(k + 1) \geq 0, \quad \Delta^2 \psi(k) \overset{df}{=} \Delta(\Delta \psi(k)) \geq 0,$$

$$\Delta^3 \overset{df}{=} \Delta(\Delta^2 \psi(k)) \geq 0, \quad k = 1, 2, \ldots .$$

Then, for all $n \in \mathbb{N}$ satisfying the condition

$$\frac{4n^{1-r}}{\alpha r} + \frac{2n}{(\alpha nr)^2} \leq 1 \quad \text{if} \quad \beta \in \mathbb{Z}$$

(2)

or the condition