NONLINEAR BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

A. A. Boichuk

We consider nonlinear boundary-value problems (with Noetherian operator in the linear part) for systems of ordinary differential equations in the neighborhood of generating solutions. By using the Lyapunov–Schmidt method, we establish conditions for the existence of solutions of these boundary-value problems and propose iteration algorithms for their construction.

1. Statement of the Problem

Consider a nonlinear boundary-value problem for systems of ordinary differential equations
\[ \dot{z} = Z(z, t), \]
\[ l_z = \varphi(z(\cdot)) \]
in the neighborhood of solutions \( z_0 = P_Q c_r, \) for every \( c_r \in \mathbb{R}^r, \) of the problem
\[ \dot{z} = 0, \quad l_z = 0, \]
which we choose as a generating problem for (1), (2). Let us find coefficient conditions for the existence of solutions \( z = z(t): z(\cdot) \in C[a, b], \) and propose iteration algorithms for their construction in the neighborhood of the solution \( z_0 = P_Q c_r \) of the generating problem (3).

We use the following assumptions and notation [1]: the nonlinear \( n \)-dimensional vector function \( Z(z, t) \) is such that \( Z(\cdot, t) \in C^1[\|z - z_0\| < \eta], \) \( Z(z, \cdot) \in C[a, b], \) \( l \) and \( \varphi \) are, respectively, linear and nonlinear \( m \)-dimensional bounded vector functionals, \( \varphi \) is continuously Fréchet differentiable, and \( Z \) and \( \varphi \) have sufficiently small Lipschitz constants.

Let \( Q = lE \) be an \( m \times n \) matrix obtained by the substitution of the identity matrix into the boundary condition (3), and let \( Q^* \) be the unique \( n \times m \) matrix pseudoinverse to \( Q \) in the Moore–Penrose sense. Denote by \( P_Q \) an \( n \times n \) matrix (an orthoprojector, i.e., \( P_Q^2 = P_Q = P_Q^* \)) that projects the real space \( \mathbb{R}^n \) onto the null space \( N(Q) \) of the matrix \( Q, \) \( P_Q: \mathbb{R}^n \rightarrow N(Q); \) similarly, \( P_Q^* \) is an \( m \times m \) matrix, \( \mathbb{R}^m \rightarrow N(Q^*). \)

Further, denote by \( P_{Qr} \) an \( n \times r \) matrix whose columns are \( r \) linearly independent columns of the matrix \( P_Q \) \( (r = n - n_1, \ n_1 = \text{rank } Q \leq \min(n, m)), \) and let \( P_{Qr}^* \) be a \( d \times m \) matrix whose rows are \( d \) linearly independent rows of the matrix \( P_Q^* \) \( (d = m - n_1). \)

The linear part of the boundary-value problem (1), (2) is an operator that does not have the inverse. Therefore, we cannot directly apply traditional methods for the investigation of boundary-value problems based on the use of the fixed-point principle. In our case, the linear part of the operator of the original boundary-value problem does not have the inverse operator because the number \( m \) of boundary conditions does not coincide with the dimension \( n \) of the differential system. Such problems for systems of ordinary differential equations are Noetherian (or with Noetherian linear part). Numerous works devoted to the investigation of problems of this type use the assumption

that these problems are Fredholm ones \( m = n \) \cite{2-4}. Moreover, in the majority of these works, it is assumed that the operator of the linear part of the original boundary-value problem has the inverse (so-called noncritical case). By using generalized inverse matrices and a version of the Lyapunov–Schmidt method \cite{1-3}, we consider critical (or resonance) boundary-value problems. Prior to the investigation of the original boundary-value problem, we formulate \cite[p. 169]{1} the criterion of solvability of the following linear inhomogeneous boundary-value problem:

\[
\begin{align*}
\dot{z} &= f_0(t), \\
    lz &= \alpha, \\
    t &\in [a, b].
\end{align*}
\]

**Lemma 1.** Suppose that \( \text{rang}(Q = I_E) = n_1 \leq \min(n, m) \). Then the inhomogeneous boundary-value problem \( (3) \) has an \( r \)-parameter family of solutions \( z_0(c_r) = P_Q f_r \). The inhomogeneous boundary-value problem \( (4) \) is solvable if and only if \( f_0(t) \in C[a, b] \) and \( \alpha \in R^m \) satisfy the condition

\[
P_Q \left\{ \alpha - l \int_a^b g(\cdot, \tau) f_0(\tau) d\tau \right\} = 0
\]

and has an \( r \)-parameter family of solutions \( z(t, c_r) : z(\cdot, c_r) \in C^1[a, b] : \)

\[
z(t, c_r) = P_Q c_r + (G[f_0])(t) + Q^* \alpha, \quad c_r \in R^r,
\]

where

\[
(G[f_0])(t) = \int_a^b g(t, \tau) f_0(\tau) d\tau
\]

is the generalized Green operator of the boundary-value problem \( (4) \) and

\[
g(t, \tau) = \left\{ \begin{array}{ll}
1, & a \leq \tau \leq t \leq b; \\
0, & a \leq t < \tau \leq b.
\end{array} \right.
\]

2. **Principal Result**

By changing the variables in \( (1), (2) \) according to the relation

\[
z(t) = z_0(c_r) + y(t),
\]

we arrive at the problem of finding a solution \( y = y(t) : y(\cdot) \in C^1[a, b] \) of the boundary-value problem

\[
\dot{y} = Z(z_0(c_r) + y, t), \\
    l y = \phi(z_0(c_r) + y(\cdot))
\]

in the neighborhood of the point \( y = 0 \) under the assumption that such a solution exists at the point \( y = 0 \). In view of the nonlinearity conditions, we have the following expansions in the neighborhood of the point \( y = 0 \):

\[
Z(z_0(c_r) + y, t) = Z(z_0(c_r), t) + A_1(t, c_r)y + R(y, t),
\]

\[
\phi(z_0(c_r) + y(\cdot)) = \phi(z_0(c_r)) + l_1 y(\cdot) + R_0(y(\cdot)),
\]