LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS ON A TORUS
THAT POSSESS GREEN–SAMOILENKO FUNCTIONS

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By using Lyapunov functions with alternating signs, we study problems of regularity and weak regularity for some linear extensions of dynamical systems on a torus.

After the introduction of the notion of Green functions of linear extensions of dynamical systems on a torus by Samoilenko [1], the problems of existence of such functions, their uniqueness and nonuniqueness, dependence on parameters, roughness, differentiability, etc. were extensively studied [2–4]. However, numerous problems still remain unsolved. The present paper is devoted to the investigation of some of these problems.

The system of differential equations

\[ \frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x, \]  

where \( \varphi \in T_m \) is an \( m \)-dimensional torus, \( x \in R^n \), \( a(\varphi) \in C_{\text{Lip}}(T_m) \), and \( A(\varphi) \in C^0(T_m) \), is called a linear extension of a dynamical system on a torus. It is customary to denote by \( \varphi_r(\varphi) \) a solution of the system \( \frac{d\varphi}{dt} = a(\varphi) \) with the initial condition \( \varphi_r(\varphi)_{t=0} = \varphi \), and \( \Omega_t(\varphi) \) denotes the matriciant of the linear system

\[ \frac{dx}{dt} = A(\varphi)(\varphi) x \]  

normalized at the point \( t = \tau \) so that \( \Omega^\tau_I = I_n \), where \( I_n \) is the \( n \)-dimensional identity matrix. Recall [2] that system (1) possesses a Green–Samoilenko function \( G_0(\tau, \varphi) \) if there exists a continuous \( n \times n \) matrix function \( C(\varphi) \) \( 2\pi \)-periodic in each variable \( \varphi_j, j = 1, m \), i.e., \( C(\varphi) \in C^0(T_m) \), such that the function

\[ G_0(\tau, \varphi) = \begin{cases} \Omega^0(\varphi)C(\varphi)(\varphi), & \tau \leq 0, \\ \Omega^\tau(\varphi)[C(\varphi)(\varphi) - I_n], & \tau > 0, \end{cases} \]  

satisfies the estimate

\[ \| G_0(\tau, \varphi) \| \leq Ke^{-\gamma|\tau|} \]  

with positive constants \( K \) and \( \gamma \) that are independent of \( \varphi \in T_m \) and \( \tau \in R \). It is clear that the choice of the norm \( \| G \| \) of a matrix is insignificant in this case. We use the operator norm \( \| G \| = \max_{\| x \|=1} \| Gx \| \), where \( \| y \|^2 = \langle y, y \rangle \) and
\[
(y, x) = \sum_{i=1}^{n} x_i y_i
\]
is the standard scalar product in \(R^n\). In what follows, we also denote \(\|S\|_0 = \max_{\varphi \in T_m} \|S(\varphi)\|\).

It is easy to show that the validity of estimate (4) is equivalent to the validity of the following estimates:

\[
\left\| \Omega_0^2(\varphi) C(\varphi) \right\| \leq K e^{-rt}, \quad t \geq 0,
\]

\[
\left\| \Omega_0^2(\varphi) [C(\varphi) - I_n] \right\| \leq K e^{rt}, \quad t \leq 0,
\]

with the same positive constants \(K\) and \(\gamma\).

We say that system (1) is regular if it has the unique Green function (3) with estimate (4). If it is known that system (1) possesses at least one Green function (3) with estimate (4), then this system is called weakly regular. If system (1) has at least two different Green functions (3) with estimate (4), it is called strictly weakly regular.

Let us formulate a criterion of weak regularity.

**Theorem 1** [2]. In order that system (1) be weakly regular, it is necessary and sufficient that there exist a quadratic form

\[
V'(\varphi, y) = \langle S(\varphi)y, y \rangle
\]

with a continuously differentiable matrix of coefficients \(S(\varphi) \in C^1(T_m)\) that has a positive-definite derivative along the solutions of the adjoint system

\[
\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dy}{dt} = -A^*(\varphi)y
\]

\((A^*(\varphi) = A^T(\varphi)\) is the transposed matrix), i.e.,

\[
V'(\varphi, y) = \left( \left( \frac{\partial S(\varphi)}{\partial \varphi} a(\varphi) - S(\varphi)A^*(\varphi) - A(\varphi)S(\varphi) \right)y, y \right) \geq \|y\|^2 \quad \forall y \in R^n.
\]

Then system (1) is regular if and only if \(\det S(\varphi) \neq 0\), and strictly weakly regular if \(\det S(\varphi_0) = 0\) for some \(\varphi = \varphi_0\).

**Remark 1.** In the case where \(\det S(\varphi) \neq 0 \quad \forall \varphi \in T_m\), the quadratic form \(V(\varphi, x) = -\langle S^{-1}(\varphi)x, x \rangle\) has a positive definite derivative along the solutions of system (1).

We give several examples of systems of the form (1) that have Green–Samoiilenko functions.

**Example 1.**

\[
\frac{d\varphi}{dt} = \sin \varphi, \quad \frac{dx}{dt} = (\cos \varphi)x.
\]

This system is strictly weakly regular because the derivative of the function \(V = -(\cos \varphi)y^2\) along the solutions of the system \(d\varphi / dt = \sin \varphi, \ dy / dt = -(\cos \varphi)y\) satisfies the estimate \(V = (\sin^2 \varphi + 2 \cos^2 \varphi)y^2 \geq y^2\) and \(\cos \pi/2 = 0\). In this case, the system

\[
\frac{d\varphi}{dt} = \sin \varphi, \quad \frac{dx}{dt} = (\cos \varphi)x
\]