INTEGRAL EQUATIONS IN THE LINEAR THEORY OF ELASTICITY IN SEMIINFINITE DOMAINS

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UDC 517.9

We investigate linear integral equations on a semiaxis that appear in the course of construction of solutions of boundary-value problems in the theory of elasticity in such domains as a semiinfinite strip or a cylinder. By using the Mellin transformation and the theory of perturbations of linear operators, we establish general results concerning the solvability and asymptotic properties of solutions of the equations considered. We give examples of application of the general statements obtained to specific integral equations in the theory of elasticity.

1. The application of the method of superposition [1, Chap. 8] to the construction of solutions of various boundary-value problems in the linear theory of elasticity in a semistrip or a semiinfinite cylinder leads to the necessity of investigation of linear integral equations on a semiaxis

\[ X(s) - \int_{0}^{\infty} Q(s, t)X(t) \, dt = F(s), \quad s > 0, \]

where the form of the kernel \( Q(s, t) \) is determined by a specific boundary-value problem and by the choice of certain systems of functions used for the construction of solutions. The general characteristic properties of these kernels can be formulated in the abstract form as follows: The function \( Q \) is representable as a sum \( Q = Q_0 + Q_1 \), the "principal" part of which is

\[ Q_0(s, t) = t^{-1} \sum_{n=1}^{\infty} \phi_1(st_n^{-1}) \phi_2(t_n^{-1}) t_n^{-1}, \]

where \( t_n = t_1 + n - 1, \ n = 1, 2, \ldots, \) for some \( t_1 \in (0, 1] \). The functions \( \phi_j \) are analytic in a certain sector of the complex plane \( \Sigma = \{ \lambda: |\arg \lambda| < \theta_0 \}, \ \theta_0 \in (0, \pi/2) \), and admit the estimates

\[ |\phi_j(\lambda)| \leq c|\lambda|^{v_j} (1+|\lambda|)^{-(v_0+v_j)}, \quad \lambda \in \Sigma, \ j = 1, 2, \]

and \( Q_1 \) is a continuous function of the variables \( s \geq 0 \) and \( t \geq 0 \) with the estimates

\[ |Q_1(s, t)| \leq cs^{v_0}(1+s)^{-(v_0+v_4)}(1+t)^{-(1+v_4)}, \quad s, t \geq 0, \]

and exponents \( v_j > 0, \ j = 1, 2, 3, 4, \) and \( v_0 \geq 0 \).

In the present paper, we investigate certain general solvability properties and asymptotic properties of solutions as \( s \to \infty \) for integral equations of the form (1)-(4). The indicated structure of the kernel \( Q \) enables us to apply the technique of the Mellin transformation and the theory of perturbations of linear operators. We also give specific examples of the application of the general statements obtained.
2. Let us introduce the functional spaces necessary for what follows. For real $\nu$ and $\mu$, denote by $B_{\nu,\mu}$ the Banach space of complex functions measurable (Lebesgue measurable) on the semiaxis $R_+ = (0, \infty)$ with the norm

$$
\|X\|_{B_{\nu,\mu}} = \sup_{t > 0} X(t)(1 + t^{-\nu})(1 + t)^{\mu}.
$$

We denote $B = B_{0,0}$, i.e., $B = L_\infty(R_+)$. Let $C_0^\infty(R_+)$ be the set of finite functions infinitely differentiable on a semiaxis with supports from $R_+$. Denote by $H_\sigma$, $\sigma \in R$, the Hilbert space of functions on $R_+$ with the norm

$$
\|X\|_{H_\sigma} = \left( \int_0^\infty |X(t)|^2 t^{\sigma-1/2} dt \right)^{1/2}.
$$

For real $\sigma$, we denote by $P_\sigma$ the straight line $\Re \gamma = \sigma$, which is parallel to the imaginary axis of the complex plane. For $-\infty < b_1 < b_2 \leq \infty$, we introduce the Hilbert space $H_2(b_1, b_2)$ [2, Chap. 7, Sec. 2] of functions analytic in the strip $\Re \gamma \in (b_1, b_2)$ with estimate uniform in $\sigma \in (b_1, b_2)$ in the norm $\|X\|_{L_2(P_\sigma)}$.

We define linear integral operators $A$ and $A_j$, $j = 0, 1$, as follows:

$$
A = A_0 + A_1, \quad (A_j X)(s) = \int_0^\infty Q_j(s,t) X(t) dt, \quad s > 0.
$$

In the notation introduced, Eq. (1) can be rewritten in the form

$$
X - AX = F,
$$

where the linear operator $A$ acts continuously in the space $B$ [see estimates (3) and (4)]. In addition to Eq. (1), we consider the "unperturbed" equation

$$
X - A_0 X = F_0.
$$

Let $I$ denote the identical transformation.

Consider the Mellin integral transformation [3]

$$
M[X](\gamma) = \int_0^\infty X(t) t^{\gamma-1} dt,
$$

which realizes an isometry (to within normalization) between the spaces $H_\sigma$ and $L_2(P_\sigma)$, with the inverse transformation

$$
X(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} M[X](\xi) t^{-\xi} d\xi.
$$

In particular, if $\mu > \nu$, $\nu + \mu > 0$, and $X \in B_{\nu,\mu}$, then $M[X] \in H_2(-\nu + \delta, \mu - \delta)$ for any $\delta \in (0, \mu - \nu)$. Consider the functions

$$
\Phi_j(\gamma) = M[\varphi_j](\gamma), \quad \Phi(\gamma) = \Phi_1(\gamma)\Phi_2(\gamma), \quad D(\gamma) = 1 - \Phi(\gamma).
$$