We consider the problem of construction of the domain of attraction for systems with aftereffect. The structure of solutions of the Cauchy problem in the vector space of states of a system is clarified with examples. We describe a constructive method for the estimation of the domain of attraction from the inside with the use of the Lyapunov functions. This method is used for the estimation of the effect of delay in a control device in the process of solution of the problem of uniaxial orientation of a spacecraft.

The well-known problem [1] about the domain of asymptotic stability for ordinary differential equations (or for difference equations) loses its well-posedness for equations with aftereffect because there is no definition of domain of attraction in this case. If a solution is sought in a functional space [2], then different domains of attraction are obtained for different norms. Recall that the notion of domain of attraction was first introduced in the theory of automatic control, which uses nonlinear differential equations, for the estimation of the spread of the variables of the state of a system under stable operation of a device.

Consider the mathematical statement of the problem in the simplest form:

\[ \dot{X}(t) = F(X(t), X(t-h)), \quad (1) \]

where \( X \in \mathbb{R}^n \), \( F(0,0) = 0 \), \( F \) satisfies the Lipschitz condition in all arguments in the neighborhood of zero, and \( h \geq 0 \) is the delay. If \( X \equiv 0 \) is asymptotically stable, then there exists an open domain \( A \subset \mathbb{R}^n \) such that, for \( h = 0 \) and any initial vector \( X_0 \in A \), the solution \( X = X(t, X_0) \to 0 \) as \( t \to \infty \) and, furthermore, \( X(t, X_0) \in A \) for any \( t \). The indicated domain \( A \) is called the domain of attraction, and its boundary can be found with the use of the Zubov theorem [1]. The introduction of a delay \( h > 0 \) changes the mathematical statement of the problem because the initial function for system (1) can be chosen as follows: Let \( \phi(t) \in A \) for any \( t \in [-h, -\frac{1}{2}(h+\delta)] \cup [-\frac{1}{2}(h-\delta), 0] \) and \( \phi(t) \) take arbitrarily large bounded values for \( t \in [-\frac{1}{2}(h+\delta), -\frac{1}{2}(h-\delta)] \). Then, for sufficiently small \( \delta > 0 \), the solution belong to \( A \). Thus, the exit from the domain \( A \) does not necessarily lead to the destabilization of the system under control.

However, if, for fixed \( h > 0 \), the trivial solution of system (1) is asymptotically stable, then there exists \( \Delta > 0 \) such that, for any initial function \( \phi(t) : \max_{t \in [-h,0]} \|\phi(t)\| < \Delta \), the solution \( X(t, \phi) \to 0 \) as \( t \to \infty \). The norm in the definition of stability is the standard norm for the space of continuous functions.

All solutions of system (1) from the \( \Delta \)-neighborhood of zero can leave it, but, for certain finite \( T \), they must return to the \( \Delta \)-neighborhood and remain there for all \( t > T \). Consequently, in \( \mathbb{R}^n \), we can define the domain of images of the \( \Delta \)-neighborhood by virtue of system (1) as follows:

\[ A(\Delta) = \left\{ X(t, \phi), t \in [0, T(\phi)], \max_{s \in [-h,0]} \|\phi(s)\| < \Delta \right\}. \quad (2) \]

Clearly, \( A(\Delta) \) includes the \( \Delta \)-neighborhood and is bounded in \( \mathbb{R}^n \). However, unfortunately, \( A(\Delta) \) can be the
range of initial functions for which solutions do not tend to zero. The following simple example shows how the structure of the domain of attraction changes with the introduction of a delay:

**Example 1.** Let

\[
\dot{x} = (x^2(t-h) - 1)x(t).
\]

If \( h = 0 \), then this equation has the asymptotically stable equilibrium position \( x = 0 \) and two unstable equilibrium positions \( x = \pm 1 \). For \( h > 0 \), by using the linear approximation, one can also establish that \( x = 0 \) is asymptotically stable and \( x = \pm 1 \) are unstable. By using the method of steps, one can construct the solution

\[
x(t) = x_0 \exp \int_{t-h}^{t} [\varphi^2(s) - 1] \, ds, \quad t \in [0, h].
\]

Hence, the following properties of the solution \( x(t, \varphi) \) are obvious:

(i) \( x(t, \varphi) \) preserves the sign of \( x_0 \);

(ii) if \( |\varphi(s)| < 1 \), then \( |x(t, \varphi)| < 1 \) for \( t \in [0, h] \);

(iii) if \( |\varphi(s)| > 1 \), then \( |x(t, \varphi)| \) increases with \( t \). Consequently, the interval \((-1, 1)\) can be regarded as the domain of attraction of zero.

However, there are solutions that take values outside the domain of attraction but tend to zero at infinity. Consider an initial function of the form

\[
\varphi(s) = \begin{cases} 
  a & \text{if } s \in [-h, 0), \\
  x_0 & \text{if } s = 0.
\end{cases}
\]

Then the solution at the first step \( t \in [0, h] \) has the form

\[
x(t) = x_0 e^{(a^2-1)t}.
\]

If \( x_0 \in (-1, 1) \), then there exists \( a > 1 \) such that

\[
|x_0| e^{(a^2-1)t} < 1, \quad t \in [0, h].
\]

Indeed, taking into account that the exponential is monotonic, we can solve the last inequality for \( t = h \) with respect to \( a \). As a result, we get

\[
1 < a < \sqrt{1 - \frac{1}{h} \ln|x_0|}.
\]

Under these conditions on the initial function, the solution remains at the second step in the interval \((-1, 1)\), etc. Thus, solutions that do not belong to the domain of attraction asymptotically tend to zero.