OPTIMAL CONTROL OVER EVOLUTION STOCHASTIC SYSTEMS
AND ITS APPLICATION TO STOCHASTIC MODELS
OF FINANCIAL MATHEMATICS

A. V. Svishchuk and A. G. Burdeinyi

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We consider problems of optimal stabilization of controlled evolution stochastic systems in semi-Markov media and their application to financial stochastic models.

1. Controlled Evolution Stochastic Systems

Consider controlled systems of the form

$$dS(t) = \mu(x(t), S(t), u)dt + \sigma(x(t), S(t), u)dw(t),$$

where $x(t)$ is a semi-Markov process, $\mu(x, s, u)$ and $\sigma(x, s, u)$ are functions continuous in the collection of variables on $X \times R \times U$, $(X, \mathcal{X})$ is a measurable phase space of $x(t)$, $U$ is a class of control, $u \in U$ is a scalar control parameter, and $w(t)$ is a Wiener process. We assume that the control $u$ in (1) depends on $S(t)$ and $x(t)$, i.e., $u \equiv u(S(t), x(t))$. Then $(S(t), x(t), \gamma(t))$ is a Markov process; here, $\gamma(t) = t - \tau_{v(t)}$, $v(t) = \max \{n: \tau_n \leq t\}$, $\tau_n = \sum_{k=1}^{n} \theta_k$, $x(t) = x_{v(t)}$, and $\{x_n, \theta_n, n \geq 0\}$ is a Markov renewal process [1].

A function $u = u(s, x)$ is called admissible if the coefficients $\mu(x, s, u)$ and $\sigma(x, s, u)$ are continuous, have first derivatives with respect to $s$, and satisfy the condition $u(0, x) = 0 \forall x \in X$. We assume that $U$ is the class of admissible controls. Every function (admissible control) $u \in U$ is associated with a process $(S^u(t), x(t))$ that is a solution of Eq. (1) with the initial conditions $S^u(0) = s$ and $x(0) = 0$.

By analogy with the deterministic case [2], we consider the following problems of stabilization by an admissible control $u \in U$:

1. Find an admissible control $u = u_0(s, x)$ such that, for $u = u_0(s, x)$, Eq. (1) is asymptotically (exponentially) stable. This is the problem of asymptotic stabilization [3, 4].

2. Find an admissible control $u = u_0(s, x)$ that minimizes the functional of the quality criterion

$$G^u_x(u) = \int_0^\infty M_x K(S^u_x(t), u(S^u_x(t), x(t))), x(t))dt,$$

where $K(s, u, x) \geq 0$, $s \in R$, $u \in U$. This is the problem of optimal stabilization in the sense of the quality criterion under consideration [3, 4].

In what follows, we assume that the quality criteria satisfy the following condition: For any $u \in U$, there exists $p > 0$ such that

$$K(s, u, x) \geq c(x)|s|^p,$$

where $c(x)$ is a positive bounded function.
Under condition (2), both problems of stabilization are closely connected, namely, if a control $u_0(s, x)$ solves problem 2 for the function $K(s, u, x)$ from (2), then

$$\lim_{t \to \infty} M_x \left| S_{x_0}^t (t) \right|^p = 0 \quad \forall x \in X.$$  

Under certain additional conditions, asymptotic and exponential stability follows from (3).

2. Bellman Principle for Evolution Stochastic Systems

In this section, we prove a theorem that is a modification of the Bellman principle for the case of problems of optimal stabilization of stochastic differential equations with semi-Markov switchings.

Let $V(s, x, t)$ be a function from the class $C^2(R \times X \times R_+)$ As is known [1], $y(t) = (x(t), t - \tau_{v(t)})$ is a Markov process in the space $Y = X \times R$ with the infinitesimal operator

$$Qf(t, x) = \frac{d}{dt} f(t, x) + \frac{g_x(t)}{G_x(t)} \left[ Pf(0, x) - f(t, x) \right] \quad \forall f(t, x) \in C^1(R_+ \times X),$$

$$G_x(t) = 1 - G_x(t), \quad g_x(t) = \frac{dG_x(t)}{dt}, \quad g_x(0) \neq 0,$$

$$G_x(t) = P \{ \theta_{n+1} \leq t \mid x_n = x \}, \quad P \text{ is the operator of transition probability } P(x, A) \text{ of the Markov chain } (x_n, n \geq 0), \quad P(x, A) = P \{ x_{m+1} \in A \mid x_n = x \}, x \in X, A \in \mathcal{F}.$$

Thus, $(S(t), y(t))$ is a Markov process in the space $R \times X \times R_+$ with the infinitesimal operator

$$L_u = \mu(x, s, u) \frac{d}{ds} + \frac{1}{2} G_x^2(x, s, u) \frac{d^2}{ds^2} + Q.$$  

Theorem 1. Suppose that there exists a positive-definite function $V_0(s, x, t) \in C^2(R \times X \times R_+)$ and a function $u_0(s, x) \in U$ that satisfy the following conditions for all $s \in R$ and $u \in U$ and certain positive constants $p, n, k_1, k_2$:

$$V_0(s, x, t) \leq k_1 |s|^p, \quad \frac{\partial V_0}{\partial s} \leq k_1 (1 + |s|^{p-1}),$$

$$L_{u_0} V_0(s, x, t) + K(s, u_0(s, x), x) \equiv 0,$$  

$$L_u V_0(s, x, t) + K(s, u(s, x), x) \geq 0,$$  

$$K(s, u, x) \geq k_2 |s|^p \quad \forall x \in X.$$

Then the function $u_0(s, x)$ is a solution of the problem of optimal stabilization of system (1) in the sense of the quality criterion $G_x^2(u)$, and

$$G_x^2(u_0) = \min_{u \in U} G_x^2(u) = V_0(s, x, 0).$$

Furthermore, the control $u_0(s, x)$ stabilizes Eq. (1) to exponential $p$-stability.