COSMOLOGICAL SOLUTION OF THE EINSTEIN–WEYL EQUATION

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The accurate integration of the Einstein–Weyl field equations is considered for the case when the spinor field depends only on the time, while the metric specifies a uniform space–time of type I in the Bianci classification, i.e., a particular case of a Steckel space of type (3.0).

INTRODUCTION

The accurate integration of a system of self-consistent Einstein–Weyl equations is one of the more complex problems in modern mathematical physics. Only a few solutions of these equations are known at present (see, for example, [1-12] and the papers cited there), in contrast, for instance, to the Einstein–Maxwell equations, for which dozens of accurate solutions have been found and investigated [13-18]. To investigate uniform spaces satisfying the Einstein–Dirac equations, we begin with the case of a massless spinor field that depends only on the time.

FIELD EQUATIONS

Consider a space of Bianci type I, with a metric of the form

$$g_{00} = 1, \quad g_{0k} = 0, \quad g_{ik} = -\gamma_{ik},$$

where $\gamma_{ij}$ is the metric of a three-dimensional space with the signature $(+, +, +)$. It is simple to establish that this space permits a three-parameter Abelian group of motions, and hence is a Steckel space of type (3.0). The orthogonal tetrad is chosen in the form

$$e_{0i} = (1, 0, 0, 0), \quad e_{(1)i} = (0, A, B, C),$$
$$e_{(2)i} = (0, K, S, V), \quad e_{(3)i} = (0, P, M, Z),$$


Using this tetrad, we construct the Newman–Penrose tetrad

$$l_{i} = \frac{1}{\sqrt{2}} (e_{0i} + e_{(1)i}), \quad n_{i} = \frac{1}{\sqrt{2}} (e_{0i} - e_{(1)i}),$$
$$m_{i} = \frac{1}{\sqrt{2}} (e_{(2)i} + ie_{(3)i}), \quad \bar{m}_{i} = \frac{1}{\sqrt{2}} (e_{(2)i} - ie_{(3)i})$$

and obtain the following relations between the spin factors

$$\lambda = -\bar{\sigma}, \quad \nu = \bar{\kappa}, \quad \pi = \bar{\tau}, \quad \gamma = -\bar{\epsilon}.$$
$$\alpha = \bar{\beta}, \quad \mu = -\bar{\rho} = \bar{\mu}, \quad \alpha = \frac{1}{2} (\bar{\tau} - \bar{\kappa}).$$
Letting \( \lambda = \frac{\partial x}{\partial N} \), we write the spin components of the Ricci tensor in the form

\[
\omega_{00} = \rho - \mu^2 - \sigma - \rho (\varepsilon + \xi) - 2 \pi \kappa + \pi \kappa + \pi \kappa.
\]
\[
\Phi_{02} = - \sigma + 2 \rho \kappa - t^2 + \kappa \sigma - (3 \pi + \varepsilon).
\]
\[
\Phi_{12} = \frac{1}{2} [\kappa \varepsilon + 3 \pi \xi - \pi \pi - \pi \pi + \pi \pi + 3 \pi \kappa + \pi \kappa + \pi \kappa],
\]
\[
(\Phi_{11} - 3 \Lambda) = - 2 (\varepsilon + \xi - \mu) - 2 p^2 - 2 \pi \pi - 2 \pi \pi + 2 \pi \kappa
\]
\[
- 2 (\varepsilon + \xi)^2 + 2 \rho (\varepsilon + \xi).
\]
\[
(\Phi_{11} + 3 \Lambda) = 2 \rho^2 + 2 \pi \pi - 2 \pi \kappa + 2 \rho (\varepsilon + \xi).
\]
\[
\Phi_{01} = - \Phi_{12}, \quad \Phi_{22} = \Phi_{00}.
\]

To obtain the Einstein–Weyl equations, we need to find the energy–momentum tensor of the two-component spinor field

\[
T_{\mu\nu'} = \varphi_{\mu'} \nabla_{\nu'} \varphi_{\mu} + \varphi_{\mu'} \nabla_{\nu} \varphi_{\mu} - \varphi_{\mu} \nabla_{\nu'} \varphi_{\mu'} - \varphi_{\mu} \nabla_{\nu'} \varphi_{\mu'}.
\]

Here

\[
\nabla_{\nu'} \varphi_{\mu} = D_{\nu \varphi_{\mu}} - e_{\nu \varphi_{\mu}} + \kappa \varphi_{\mu},
\]
\[
\nabla_{\nu} \varphi_{\mu} = D_{\nu \varphi_{\mu}} - e_{\nu \varphi_{\mu}} + \kappa \varphi_{\mu},
\]
\[
\nabla_{\nu'} \varphi_{\mu'} = D_{\nu \varphi_{\mu'}} - e_{\nu \varphi_{\mu'}} + \kappa \varphi_{\mu'},
\]
\[
\nabla_{\nu} \varphi_{\mu'} = D_{\nu \varphi_{\mu'}} - e_{\nu \varphi_{\mu'}} + \kappa \varphi_{\mu'},
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\nabla_{\nu} \varphi_{\mu'} = D_{\nu \varphi_{\mu'}} - e_{\nu \varphi_{\mu'}} + \kappa \varphi_{\mu'},
\]
\[
\nabla_{\nu'} \varphi_{\mu} = D_{\nu \varphi_{\mu}} - e_{\nu \varphi_{\mu}} + \kappa \varphi_{\mu}.
\]

The spinor field of spatial rotation of the tetrad may be diagonalized and made real (one real component remains). For this case, the energy–momentum tensor takes the form

\[
T_{\mu\nu'} = 2 \kappa \beta_{\mu'} (\varepsilon - \pi),
\]
\[
T_{\mu\nu'} = i \kappa \beta_{\mu'} \varepsilon,
\]
\[
T_{\mu\nu'} = -2 \kappa \beta_{\mu'} \sigma, \quad T_{\mu\nu'} = i \kappa \beta_{\mu'} \kappa,
\]
\[
T_{\mu\nu'} = 0, \quad T_{0111'} = i \kappa \beta_{0111'} (\varepsilon - \pi), \quad T_{a110'} = 0.
\]

The field equations take the form

\[
\rho - \mu^2 - \sigma \pi - 2 \rho \pi \kappa - 4 \pi \kappa = 0;
\]