COSMOLOGICAL SOLUTION OF THE EINSTEIN–WEYL EQUATION

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The accurate integration of the Einstein–Weyl field equations is considered for the case when the spinor field depends only on the time, while the metric specifies a uniform space–time of type I in the Bianci classification, i.e., a particular case of a Steckel space of type (3.0).

INTRODUCTION

The accurate integration of a system of self-consistent Einstein–Weyl equations is one of the more complex problems in modern mathematical physics. Only a few solutions of these equations are known at present (see, for example, [1-12] and the papers cited there), in contrast, for instance, to the Einstein–Maxwell equations, for which dozens of accurate solutions have been found and investigated [13-18]. To investigate uniform spaces satisfying the Einstein–Dirac equations, we begin with the case of a massless spinor field that depends only on the time.

FIELD EQUATIONS

Consider a space of Bianci type I, with a metric of the form

\[ g_{00} = 1, \quad g_{0\alpha} = 0, \quad g_{ij} = -\gamma_{ij}, \]

where \( \gamma_{ij} \) is the metric of a three-dimensional space with the signature (+, +, +). It is simple to establish that this space permits a three-parameter Abelian group of motions, and hence is a Steckel space of type (3.0). The orthogonal tetrad is chosen in the form

\[ e_{(0)\lambda} = (1, 0, 0, 0), \quad e_{(1)\lambda} = (0, A, B, C), \]
\[ e_{(2)\lambda} = (0, K, S, V), \quad e_{(3)\lambda} = (0, P, M, Z), \]


Using this tetrad, we construct the Newman–Penrose tetrad

\[ l_i = \frac{1}{\sqrt{2}} (e_{(0)i} + e_{(1)i}), \quad n_i = \frac{1}{\sqrt{2}} (e_{(0)i} - e_{(1)i}), \]
\[ m_i = \frac{1}{\sqrt{2}} (e_{(2)i} + ie_{(3)i}), \quad \bar{m}_i = \frac{1}{\sqrt{2}} (e_{(2)i} - ie_{(3)i}) \]

and obtain the following relations between the spin factors

\[ \lambda = -\bar{\sigma}, \quad \nu = -\bar{\kappa}, \quad \pi = \bar{\tau}, \quad \tau = -\bar{\epsilon}. \]
\[ \alpha = -\bar{\beta}, \quad \mu = -\rho = \bar{\mu}, \quad \bar{\alpha} = \frac{1}{2} (\bar{\tau} - \bar{\kappa}). \]
Letting $\chi = \partial_\beta x N/2$, we write the spin components of the Ricci tensor in the form

$$\omega_{00} = \rho - \mu^2 - \sigma \sigma - (e + e) - 2\kappa \kappa + \kappa \kappa,$$

$$\omega_{02} = -\sigma + 2\rho \sigma - \sigma^2 + \sigma (3\varepsilon - e),$$

$$\omega_{12} = \frac{1}{2} \left[ \kappa - e + 3\tau \rho - e + \kappa + \kappa + 3\kappa \kappa + \sigma \kappa + \sigma \kappa \right].$$

\begin{equation}
(\phi_{11} - 3\lambda) = -2(e + e - \mu) - 2\rho^2 - 2\sigma^2 - 4\tau - 2\kappa - 2e - 2\kappa - 2(e + e),
\end{equation}

$$\phi_{01} = -\phi_{12}, \quad \phi_{22} = -\phi_{00}.$$

To obtain the Einstein–Weyl equations, we need to find the energy–momentum tensor of the two-component spinor field

$$T_{\alpha\beta} = \frac{\partial}{\partial x_\alpha} \psi_\beta + \frac{\partial}{\partial x_\beta} \psi_\alpha - \psi_\alpha \frac{\partial}{\partial x_\alpha} \psi_\beta - \psi_\beta \frac{\partial}{\partial x_\beta} \psi_\alpha.$$

Here

$$\nabla_{\alpha} \psi_\beta = D\psi_0 - e\psi_0 + \kappa \psi_1,$$

$$\nabla_{\alpha} \psi_1 = D\psi_1 - e\psi_1 + \psi_0 \psi_0 + \psi_0 \psi_0 + \psi_0 \psi_0 + \psi_0 \psi_0 + \psi_0 \psi_0 + \psi_0 \psi_0 + \psi_0 \psi_0 + \psi_0 \psi_0.$$