COVARIANT SPINOR DERIVATIVE ASSOCIATED WITH
ARBITRARY LINEAR CONNECTEDNESS

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A covariant spinor derivative is constructed in arbitrary reference frames and also in coordinates, with
specified arbitrary linear connectedness.

INTRODUCTION

Constructing a symmetric energy–momentum tensor for spinor fields on a covariant path entails considering spinors
in arbitrary nonorthogonal reference frames [1]. This involves constructing the covariant derivative of spinors in arbitrary
reference frames. In the present work, we solve the more general problem of constructing the covariant spinor derivative
associated with arbitrary linear connectedness in a Riemann manifold.

COVARIANT SPINOR DERIVATIVE IN ORTHOGONAL REFERENCE FRAMES

The covariant derivative of a spinor was first obtained in [2], on the basis that bilinear spinor combinations $A_i =
\psi \alpha_i \psi$, where $\alpha_0 = I$, $\alpha_i = \gamma^5 \gamma^i / \gamma^0$ ($i = 1, 2, 3$), $\alpha_4 = -i \gamma^2$, and $\alpha_5 = \gamma^0 \gamma^5$, form one 4-vector ($i = 0, 1, 2, 3$) and two
invariants $i = 4, 5$, for which the law of parallel transfer with infinitely small transformations is known. Assuming that the
increment in the spinor in such transformations is proportional to the initial spinor with some proportionality matrix and the
differential of the displacement $\delta \psi = \sum \epsilon_\ell C_\ell d\psi$, where $C_\ell$ is the proportionality matrix and $d\psi$ are orthogonal components of
the displacement (implicitly assuming the Leibnitz rule), and comparing the transformation laws for $A_i$ with the transforma-
tion law for the vector and invariant, we obtain a system of matrix equations for the proportionality matrix

$$C_\ell^+ a_i + a_i C_\ell = \sum \epsilon_\ell \Gamma_{i\ell}(a_i),$$

when $i = 0 \rightarrow 3$, and $C_\ell^+ a_i + a_i C_\ell = 0$, when $i = 4, 5$. The general solution of these equations includes not only the well-
known Fock connectedness, but also an additive matrix proportional to the unit matrix, which was interpreted by Fock as the
potential of the electromagnetic field. This arbitrariness in the definition of the covariant spinor derivative is eliminated by
imposing the requirement that the covariant derivative of the metric spinor in spinor space must be zero [3].

A subtle method of constructing the covariant derivative of spinors was proposed in [4]. In this approach, together
with stratification $O(V)$ of the orthogonal reference frames by $V$, the fundamental stratification $S(V)$ of the spinor reference
frames, for which the structural group Spin(4) is a representation of the Lorentz group in the space of contravariant spinors
of rank 1, is considered. Expansion of $O(V)$ yields $S(V)$. Connectedness determined by the Christoffel symbols from the
metric tensor exists in the Riemann manifold, and is called Riemann connectedness. The connectedness in stratification of the
linear reference frames is called linear connectedness, and that in stratification of the orthogonal reference frames is called
Lorentz connectedness. Lorentz connectedness is Riemann connectedness if and only if the linear connectedness associated
with the Lorentz connectedness has zero curvature and the 1-form $\omega = (\omega_\mu)\epsilon^\mu$ with specified cross section $\epsilon_\mu(x)$ for the
principal stratification $O(V)$ is
where \(c_{abc}\) are the Ricci rotation coefficients; \(\xi^a\) is the vector field. This form of connectedness for \(O(V)\) may be related to the spinor connectedness \(\sigma\) for \(S(V)\) by the law \(\sigma(V_x) = p^{-1}\omega(pV_x)\), where \(p(V_x)\) is the projection of the tangential vector \(V_x\) in \(O(V)\) onto \(z\) in \(S(V)\), and \(p'\) is an isomorphism of the Lie algebra of the group \(\text{Spin}(4)\) onto the Lie algebra of the Lorentz group \(L(4)\). After introducing some local cross section in \(S(V)\), the form of connectedness \(\sigma\) may be written as

\[
\sigma = \frac{1}{4} \omega^a \tau_a \tau^b.
\]

If \(M\) is some four-dimensional complex vector space on which \(\text{Spin}(4)\) acts, the contravariant 1-spinor \(\psi\) at the point \(x \in V\) is defined as the mapping \(z \rightarrow \psi(z)\) from \(\pi^{-1}(x)\) to \(M\) such that

\[
\psi(zA^{-1}) = A\psi(z) (A \in \text{Spin}(4)).
\]

Finally, the covariant derivative of the spinor \(\xi^a \nabla \psi\) in the direction of the vector \(\xi^a\) is found as

\[
\xi^a \nabla \psi = \xi^a \partial_a \psi + \frac{1}{4} \omega^a \tau_a \tau^b \psi.
\]

Henceforward, following [5], the spinors are regarded as elements of spinor stratification associated with the principal stratification \(O(V)\) of the orthogonal reference frames. The associated stratification with a standard layer \(M\) on which the group \(\text{Spin}(4)\), as a representative of the group \(L(4)\), acts from the right side

\[
L \in L(4),\quad L : M \rightarrow M,\quad \psi \rightarrow \Lambda^{-1}(L)\psi = \psi \cdot L,
\]

is determined as the factor-space of the tensor product \(O(V) \times M\) relative to the action of group \(L(4)\). Suppose that, for each \(x \in V\), a representative of the equivalence class is chosen in the form \((x, L(x), \psi)\), where \(L(x)\) specifies some field of reference vectors, \(\psi \in M\). If \(\psi\) travels over \(M\), then \((x, L(x), \psi)\) will be a representative of the layer \(M_x\) at point \(x\). Each element \(z = (x, L_1) \in O(V)\) generates a mapping

\[
M \rightarrow M_x,\quad \psi \rightarrow (x, L(x), \psi \cdot (L^{-1}_1 L(x))).
\]

The curve

\[
(0 \rightarrow (x(0), L(x(0)) (1 - t\omega(\xi)) \cdot \psi(x) = (x, L(x(t)), \psi \cdot (1 + t\omega(\xi))) = (x, L(x(t)), (1 - \frac{t}{4} \omega^a \tau_a \tau^b) \psi
\]

is the horizontal path in the associated stratification and determines the parallel transfer \(\psi(x)\) from point \(x\) to point \(x(t)\). Therefore, the covariant derivative \(\nabla_\xi \psi\) in the direction of the vector \(\xi\) is

\[
\nabla_\xi \psi = \lim_{t \to 0} \frac{\psi(x(t)) - \psi(x) \cdot (1 + t\omega(\xi))}{t} = \xi^a \partial_a \psi + \frac{1}{4} \omega^a \tau_a \tau^b \psi.
\]

We obtain the result in Eq. (1).

**COVARIANT SPINOR DERIVATIVE IN ARBITRARY REFERENCE FRAMES**

Suppose that \(E(V)\) is the principal stratification of the linear reference frames with connectedness \(\omega\) corresponding to some reference field \(e_\alpha(x) = (e_\alpha^{\alpha}(x)), x \in V\). An arbitrary reference field is characterized by the cross section \(z(x) = (x, A(x)), A(x) \in GL(4)\). If parameterization of the elements of the layer of linear reference frames at the given point begins with