SOME STRANGE PROPERTIES OF THE LOGISTIC EQUATION DEFINED WITH \( r \) AND \( K \): INHERENT DEFECTS OR ARTIFACTS?

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SUMMARY

In some situations the logistic equation in the usual expression, \( \frac{dN}{dt} = r(1 - N/K)N \), exhibits properties that are biologically unrealistic. For example, when \( r \leq 0 \) the population can no longer show any normal, negative response in per-capita growth rate to increasing density. Also, when the equation is employed in the Volterra's competition model, a familiar but incredible conclusion is derived which says that the outcome of competition is entirely independent of the reproductive potential \( r \) of each species. It is shown that all such strange properties are mere artifacts arising peculiarly in this \( r-K \) model from its misleading implicit supposition that \( K \) could be independent of \( r \), and they can be readily removed by alternative use of a plainer, classical form of the model, \( \frac{dN}{dt} = (r - hN)N \).

KEY WORDS: Logistic equation, carrying capacity, competition theory, \( K \)-selection

INTRODUCTION

In the ecological literature the logistic population growth equation usually appears in the form

\[
\frac{dN}{dt} = r \left(1 - \frac{N}{K}\right)N
\]

(1)

where \( r \) and \( K \) are defined as the intrinsic rate of natural increase and the carrying capacity of the environment, respectively (e.g., Begon et al., 1990).

However, it has been occasionally noticed that in some situations this familiar model exhibits properties that are biologically unrealistic. The purpose of this note is to recall attention to this fact, which seems to have often been overlooked by ecologists, by elucidating its theoretical details and causes. The analysis may be crucial for proper use of the logistic equation as a basic model in ecology.
Results

The first point to be noted is the limitation of eq. (1) with respect to the parameter $r$. Consider, e.g., the case that the intrinsic birth- and death rates are exactly balanced, i.e. $r = b - d = 0$. Then, since $\frac{dN}{dt} = 0$ for any value of $N$, the population in eq. (1) is to remain at its initial level $N_0$, however high might it be. This evidently contradicts the original premise of density dependence in logistic population growth. The outcome when $r < 0$ is more absurd as first pointed out by Hutchinson (1978; p. 4–5) and analyzed by Fulda (1981), producing positive correlation between $(\frac{dN}{dt})/N$ and $N$.

To elucidate this further, let us examine the model’s behaviour when $r < 0$, based on the solution, $N = N_0 \left\{ (1 - N_0/K) e^{-rt} + N_0/K \right\}$, of eq. (1). As seen in Fig. 1, it generates growth patterns which are biologically nonsensical. Namely, if the initial density $N_0$ is higher than the carrying capacity $K$, the population increases until it “explodes” towards infinity at time $t = \log \left\{ N_0/(N_0 - K) \right\}/|r|$, and if $N_0$ is somehow lower than $K$, it begins to decrease with a gradually increasing rate. Thus, in addition to $K > 0$, the condition $r > 0$ proves to be an indispensable one for eq. (1) to be valid ecologically, notwithstanding that parameter $r$ originally can take any value from negative to positive.

It is also known that even when $r > 0$ the model exhibits an odd behaviour if $r$ is varied for $N > K$, a higher $r$ resulting in a higher descending rate to equilibrium (Pollard, 1981).

Furthermore, a strange situation occurs when eq. (1) is employed in the

![Fig. 1. Unrealistic population behaviours shown by the r-K logistic model with a negative r (r = -1; K = 100; N_0 = 95, 100, 105). The broken line indicates the time for population explosion for N_0 = 105.](image-url)