DIAMETRAL THEORY OF ALGEBRAIC SURFACES AND GEOMETRIC THEORY OF INVARIANTS OF GROUPS GENERATED BY REFLECTIONS. II

V. F. Ignatenko

UDC 514

We systematically present the basic principles of the geometric theory of invariants of infinite groups generated by skew reflections with respect to hyperplanes in the real Euclidean space.

Introduction

Let \( F_n \) be an \((m-1)\)-dimensional algebraic surface of order \( n \) in the real Euclidean space \( E^m \). The diametral theory of surfaces \( F_n \) was considered in [1]. This theory has various applications. The original ideas of its application to the investigation of surfaces \( F_n \) with planes of symmetry were formulated by Italian geometers Piazzola-Beloch, Rosina, and others [2-4]. Their results suggest the possibility of applying the diametral theory to the investigation of the structure of surfaces \( F_n \) with infinite groups \( G_\mu \) generated by skew (in particular, orthogonal) mappings with respect to the planes. These groups define groups of symmetries in pseudo-Euclidean spaces, and the construction of such groups enables one to consider, in particular, problems of mathematical physics related to the symmetry indicated above [5].

It should be noted that the theory of algebras of invariants \( J^G_\mu \) of groups \( G_\mu \) is rather complicated: there exist nonfree algebras \( J^G_\mu \) (Zalesskii [6]); the construction of the Nagata example is applicable for finding an infinitely generated algebra \( J^G_\mu \) (Veles'ko [7]). On the other hand, these examples encourage one to clarify still further the structure of groups \( G_\mu \) and algebras of their invariants.

The fundamentals of the geometric theory of invariants of groups \( G_\mu \) were developed by the author. This paper is devoted to the systematic presentation of its principles.

1. Rosina Lemma

Assume that a plane curve \( C_\mu \) (\( C_\mu = F_n \) in \( E^2 \)) is invariant with respect to the infinite group \( G_2 \) (\( \mu = 2 \)).

Lemma 1 (Rosina [3]). If a curve \( C_\mu \), \( n > 2 \), that is invariant with respect to the group \( G_2 \) does not consist of parallel straight lines, then it decomposes over the field of complex numbers into conics with common symmetry. In this case, in the equation of the imaginary conic, only the free term can be regarded as nonreal.

In [3], the diametral theory of algebraic curves (see also [4]) was used in the proof of this lemma. We give another proof of this statement [8] (it will be used in what follows).

Proof. Since \( C_\mu \) does not decompose into parallel straight lines, each direction of symmetry is associated with a single axis. The number \( n = 2s \) is even because there exist directions of symmetry that are not asymptotic for \( C_\mu \) [1]. Let \( a \) and \( b \) be mutually nonconjugate axes of symmetry of \( C_\mu \) and let \( s_a \) and \( s_b \) be reflections with respect to these axes. If \( A = a \cap b \), then \( s_\mu = s_b s_a \) is elliptic or hyperbolic rotation around a point \( A \). By virtue of the fact that \( G_2 \) is infinite, the number of ideal points of the curve \( C_\mu \) is equal to two (they are \( s \)-tuple). Therefore,
the leading form of the equation for $C_n$ in the Cartesian coordinates has the form $c(x_1^2 - z^2 x_2^2)$, where $z = \alpha + \beta \varepsilon (\alpha^2 + \beta^2 > 0, \alpha \beta = 0, \varepsilon = \sqrt{-1})$. The conic $\kappa$ with the equation $x_1^2 - z^2 x_2^2 = c_0$ is invariant with respect to the group $G_2'$ that is isomorphic to $G_2$. The directions of symmetry of parallel axes of the curves $\kappa$ and $C_n$ coincide.

We assume that the origin of coordinates $O \neq A$ and $0 \neq c_0 = \tilde{x}_1^2 - z^2 \tilde{x}_2^2$, where the point $(\tilde{x}_1, \tilde{x}_2) \in C_{\pi}$. If $s_a$ and $s_b$ generate an infinite group (this, in particular, takes place for $\beta = 0$), then the curves $\kappa$ and $C_n$ have more than $2n$ common points. According to the Bézout theorem, the conic $\kappa$ is contained in $C_{\pi}$. If $a$ and $b$ determine the wall of the chamber for a finite (dihedral) group $[N]$ of order $2N$, then $N > 2$, $\alpha = 0$, and the conic $\kappa$ is an ellipse because $a$ and $b$ are mutually nonconjugate. A certain centroaffine transformation $f$ maps the ellipse $\kappa$ into a circle. In this case, $G_2'$ turns into the group generated by orthogonal reflections; $f(C_n)$ has only axes of symmetry that must pass through the point $O$ ($G_2 = G_2'$). By virtue of the Bézout theorem, the curve $f(C_n)$ and, hence, the curve $C_n$ decompose into conics of the elliptic type with the same symmetry.

In the case $a \parallel b$, the transformation $s_a s_b$ is a parabolic rotation. The orbit of a point $b \in C_n$ with respect to $G_2$ and the parabola $\kappa \ni B$ with axes $a$ and $b$ contain infinitely many common points and $\kappa \ni C_n$.

The diameters of each conic (component $C_n$) are determined by the real coefficients of its equation except the free term. Note that the curve $C_n$ may have no real points. The lemma is proved.

Lemma 1 implies the following statement:

**Corollary.** The invariance with respect to $G_2$ of an algebraic curve irreducible over the field of complex numbers is a characteristic property of a conic (ellipse, hyperbola, or parabola).

In the proof of Lemma 1, we have used the idea of the proof of the equality in [9], which is based on the Bézout theorem. Among new applications of the Bézout theorem, we can mention the paper of Sabitov [10], in which an algorithm for the verification of bending of a polyhedron is described.

2. Structure of an Orbit of Directed Symmetry

An infinite set $B_{\mu}$ of planes of symmetry of a surface $F_n$ with group $G_{\mu}$ is associated with the set $N_{\mu}$ of directions of symmetry. Let $N_{\lambda} \subseteq N_{\mu}$ be an infinite $G_{\mu}(\bar{u})$-orbit of a vector $\bar{u} \in N_{\mu}$ (this concept was first introduced in [11]). In the Cartesian coordinate system $Ox_i, i = \bar{l}, \bar{m}$, the $\lambda$-plane $\Pi^\lambda(x_\bar{k}) \subseteq \Pi^\mu(x_r), k = \bar{l}, \bar{\lambda}, r = \bar{l}, \bar{\mu}$; it is associated with a set $B_{\lambda} \subseteq B_{\mu}$. Reflections with respect to the planes $B_{\lambda}$ determine the group $G_{\lambda} \subseteq G_{\mu}$. The structure of the set $N_{\lambda}$ is described by the following lemma:

**Lemma 2** [12]. **The linear span** $\Pi^\lambda$ **of the set** $N_{\lambda}$ **of the surface** $F_n$ **is the sum of 2-planes such that every 2-plane that is parallel to any 2-plane in this sum and does not lie on** $F_n$ **intersects it along real or, possibly, imaginary conics with common symmetry.**

**Proof.** Assume that the surface $F_n$ is given by the equation

$$\sum_{s=0}^{n} \varphi_{n-s}(\bar{x}) = 0,$$

where $\varphi_{n-s}(\bar{x})$ are forms of the coordinates of the vector $\bar{x} = (x_i)$ of order $n - s$. The asymptotic cone $K_n$ of the surface $F_n$ without parallel planes of symmetry is invariant with respect to the group $\tilde{G}_\lambda$ that is isomorphic to