IMPROVED SECOND ORDER DESIGN AND
THE DATUM CHOICE

Abstract

It is shown that also in a rank deficient Gauss–Markov model higher weights of
the observations automatically improve the precision of the estimated parameters as long
as they are computed in the same datum. However, the amount of improvement in terms
of the trace of the dispersion matrix is minimum for the so-called “free datum” which
corresponds to the pseudo-inverse normal equations matrix. This behaviour together
with its consequences is discussed by an example with special emphasis on geodetic
networks for deformation analysis.

0. Introduction

In the usual set-up of a Gauss–Markov Model with no rank defect namely
\[
E \{ y \} = A \xi = y - u , \quad \text{rk} \ A = m ,
\]
\[
D \{ y \} = \sigma^2 P^{-1} = D \{ u \} \quad \text{positive-definite} ,
\]
we introduce the \( n \times 1 \) vector \( y \) of observation increments, the \( m \times 1 \) vector \( \xi \) of
fixed unknown parameters, the \( n \times m \) coefficients matrix \( A \) of full column rank, the
\( n \times 1 \) vector \( u \) of observational errors, and the positive-definite \( n \times n \) weight matrix
\( P \) here \( E \) denotes “expectation” and \( D \) “dispersion”, respectively. Then by least-
squares approximation we obtain the Best Linear Uniformly Unbiased Estimate (BLUUE)
of \( \xi \) uniquely from the normal equations system
\[
N \hat{\xi} = z , \quad [ N , z ] := A^T P \{ A , y \} ,
\]
with its dispersion (or variance-covariance) matrix
\[
D \{ \hat{\xi} \} = \sigma^2 N^{-1} = \sigma^2 (A^T P A)^{-1}
\]
since \( N \) is regularly invertible. Now if some of the measurements are performed with
higher accuracy, thus leading to an increased weight matrix \( P + \delta P \) with \( \delta P \geq 0 \) in
the Löwner partial ordering (W. Marshall & I. Olkin, 1979), (i.e. \( \delta P \) is positive
semidefinite), we certainly arrive at improved estimates \( \hat{\xi}_{\text{imp}} \) according to

which is due to the identity

\[(N + \delta N)^{-1} = N^{-1} - N^{-1} \delta N (I_m + N^{-1} \delta N)^{-1} N^{-1}\] (0.5)

with

\[\delta N (I_m + N^{-1} \delta N)^{-1} = [I_m + (\delta N) N^{-1}]^{-1} \delta N \geq 0\] (0.6)

being positive-semidefinite, at least; \(I_m\) denotes the \(m \times m\) identity matrix. Note that the strict inequality sign may be used everywhere if \(\delta P > 0\) thereby indicating even positive-definiteness. In the following we investigate the analogous behaviour of higher weighting \(\delta P \geq 0\) in case of the Singular Gauss-Markov Model where both the coefficients matrix \(A\) and the weight matrix \(P\), may be rank deficient.

1. Higher weights and the datum choice

Now let us assume the Singular Gauss-Markov Model

\[E \{y\} = A \hat{\xi} = y - u, \text{ rk } A = q \leq m,\] (1.1a)

\[D \{y\} = \sigma^2 P^* = D \{u\} \text{ positive-semidefinite},\] (1.1b)

where \(P^*\) denotes the "pseudo-inverse" of the singular weight matrix \(P\), defined by

\[P P^* P = P, P^* P P^* = P^* \text{ (i.e. reflexive g-inverse)},\] (1.2a)

\[P P^* \text{ and } P^* P \text{ symmetric (i.e. orthogonal projections)}.\] (1.2b)

Under the additional restriction

\[\text{rk } N = \text{rk } A^T P A = \text{rk } A = q\] (1.3a)

which simply means that we can make full use of all the observations, and which has the equivalent form

\[A N^* N = A \text{ for any arbitrary g-inverse } N^*\] (1.3b)

again by least-squares approximation we are led to the now singular normal equations system

\[N \hat{\xi} = z, [N, z] = A^T P [A, y],\] (1.4)

which yields the unique Best Linear Uniformly Minimum Bias Estimate (BLUMBE) of \(\xi\) through

\[\hat{\xi} = N^* z \text{ (} N^* \text{ pseudo-inverse of } N)\] (1.5a)