SIMPLIFIED FORMULAS FOR GEOID HEIGHT EVALUATION

Abstract

A non-conventional treatment of Stokes' integral enables significant simplification of formulas for both the regional and global contributions of the gravity field to the geoidal height.

1. Introduction

Stokes' integral (spherical approximation)

\[ N = \frac{R}{4\pi\gamma} \int_\sigma \Delta g S(\Psi) \, d\sigma \]  

is generally used for geoid height evaluation. Here \( \Delta g \) means the gravity anomaly, \( \sigma \) denotes the unit sphere, \( R \) is the mean Earth's radius, \( \gamma \) is the mean gravity, and \( S(\Psi) \) represents Stokes' function:

\[ S(\Psi) = \frac{1}{\sin \frac{\Psi}{2}} - 3 \cos \Psi \ln \left[ \sin \frac{\Psi}{2} \left( 1 + \sin \frac{\Psi}{2} \right) \right] - 6 \sin \frac{\Psi}{2} + 1 - 5 \cos \Psi . \]  

The latter can be expanded in Legendre series

\[ S(\Psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos \Psi) . \]  

The expression (1) is customarily split into two parts:

\[ N = N^{(0)} + \delta N , \]  

\[ N^{(0)} = \frac{R}{4\pi\gamma} \int_{\sigma_o} \Delta g S(\Psi) \, d\sigma , \]  

\[ \delta N = \frac{R}{4\pi\gamma} \int_{\sigma - \sigma_o} \Delta g S(\Psi) \, d\sigma . \]  

By \( \sigma \) a spherical cap of radius \( \Psi_0 \) is denoted, centered at a computation point. The truncated Stokes' function \( N^{(0)} \) represents the inner zone contribution to \( N \) and \( \delta N \) characterizes the remote zone influence. The former is determined on the basis of gravity data while the latter is evaluated from the global geopotential models.

The theory of evaluating the remote zone influence was first developed in pioneer investigations of Molodensky, starting from his paper (1945). Many authors elaborated various modifications of Molodensky's method aiming at increasing the accuracy of geoidal height computation. The review of Pishchukhina (1987) contains, in particular, references to about 60 relevant papers.

Molodensky introduced an auxiliary function

\[
S^*(\Psi) = \begin{cases} 
0, & 0 \leq \Psi \leq \Psi_0 \\
S(\Psi), & \Psi_0 \leq \Psi \leq \pi 
\end{cases}
\]  

(7)

which allowed to write (6) as

\[
\delta N = \frac{R}{4\pi\gamma} \int_\sigma \Delta g \ S^*(\Psi) \ d\sigma .
\]

(8)

The kernel function \( S^*(\Psi) \) was presented as Legendre series

\[
S^*(\Psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n(\Psi_o) P_n(\cos \Psi) 
\]

(9)

where Molodensky's truncation coefficients are

\[
Q_n(\Psi_o) = \int_{\Psi_o}^{\pi} S(\Psi) P_n(\cos \Psi) \sin \Psi \ d\Psi .
\]

(10)

From (8), (9) and the expansion \( \Delta g = \sum_{n=2}^{\infty} \Delta g_n \) follows

\[
\delta N = \frac{R}{2\gamma} \sum_{n=2}^{n_o} Q_n(\Psi_o) \Delta g_n + \Delta N
\]

(11)

where \( \Delta N \) is the so called truncation error caused by the lack of the potential coefficients beyond degree \( n_o \). It is determined by the formula

\[
\Delta N = \frac{R}{2\gamma} \sum_{n=n_o+1}^{\infty} Q_n(\Psi_o) \Delta g_n .
\]

(11*)

Formulas (4), (5) and (11) define the geoidal height.

Different methods were proposed to calculate \( Q_n \), all of them being rather cumbersome, however, due to the complexity of Stokes' function entering (10). The most simple formulas seem to be those provided by Hagiwara (1976). An alternative representation of the geoidal height (a surface layer potential) was utilized in