ANALYTICAL COLLOCATION WITH KERNEL FUNCTIONS

Abstract

T. Krarup proposed the use of collocation with kernel functions for the approximation of a potential function on the earth surface as well as in local regions of a sphere. Starting from the smoothing criterion of the least norm of the horizontal gradients on a sphere, a neighbourhood criterion was derived taking into account smoothness as well as stability properties of the series evaluation. It is finally shown how to choose the kernel functions in order to obtain a smooth interpolation function at the surface of the earth.

1. Collocation and interpolation with harmonic kernel functions

It is often required in geodesy to compute a potential function $T$ in the space $\Omega$ outside a sphere $\omega$ with radius $r_Q$ from its boundary values $t$ at the boundary $\omega$, that is, to solve a boundary value problem for Laplace' equation $\Delta T = 0$. Due to special practical circumstances solutions in form of a series evaluation are favoured. Two kinds of solutions may be principally distinguished: the approximation can be performed using an orthogonal function system or a non-orthogonal function system. Of course, an approximation using orthogonal basic functions are usually preferred, but under some circumstances non-orthogonal basic functions are more convenient. Such a set of basic functions, non-orthogonal with respect to integration over the sphere, are the kernel functions as used in least-squares collocation (Krarup, 1969).

The application of collocation with kernel functions (series of kernel functions) was proposed by T. Krarup for two reasons:

a) in contrast to the use of a series of spherical harmonics, series of kernel functions may be well suited for the approximation of the potential in local regions on the sphere;

b) series of kernel functions may be well suited to approximate the potential on and outside the earth surface.

We will restrict ourself in the next sections to the first problem, viz. the function $t$ shall be approximated by the values $t(Q_i)$ in a grid of sample points on the sphere using collocation with kernel functions. Such a procedure will be needed e.g. for the representation of an astro-gravimetric geoid of the Federal Republic of Germany.

The potential function outside a sphere has a discrete spectrum. The spherical
harmonics as the eigenfunctions of the Laplace operator for this domain $\Omega$ form an orthogonal system of functions over the sphere. A series evaluation of spherical harmonics can be considered as a series evaluation in the spectral range of the potential function. Using the abbreviation

$$R_n^m (\theta, \lambda) = \begin{cases} \tilde{R}_{nm} (\theta, \lambda) & \text{for } m > 0 \\ \tilde{S}_{nm} (\theta, \lambda) & \text{for } m < 0 \end{cases}$$

$\tilde{R}_{nm}$, $\tilde{S}_{nm}$ normalized spherical harmonics,
$\theta$, $\lambda$ spherical coordinates,
we obtain for a continuous function $t$ on the sphere $\omega$ the spectral coefficients

$$a_n^m = \frac{1}{4\pi} \int_{\sigma} t(\theta, \lambda) R_n^m (\theta, \lambda) \, d\sigma$$

$\sigma$ surface of unit sphere
as well as for the harmonic function $T$ outside the sphere $\omega$ the spectral series

$$T(r, \theta, \lambda) = \sum_{n=0}^{\infty} (\frac{Q}{r})^{n+1} \sum_{m=-n}^{n} a_n^m R_n^m (\theta, \lambda).$$

By contrast to the application of spherical harmonics, collocation with kernel functions can then be considered as a series evaluation using a non-orthogonal system of basic functions. Collocation (as a solution of a boundary value problem) is obviously always connected with an interpolation procedure on the boundary surface; the potential function $\widetilde{T}$ must be computed in such a way that it attains fixed values $t_i$ at certain points $Q_i$ on the boundary.

The solution of an equation system

$$\sum_{i=1}^{I} b_i K(Q_i, Q_j) = t(Q_j), (j = 1, 2 \ldots)$$

$K(Q_i, Q_j)$ kernel matrix
leads to the interpolation formula (Meschkowski, 1962)

$$\tilde{t}(P) = \sum_{i=1}^{I} b_i K(P, Q_i),$$

where $K(P, Q_i)$ is a kernel function* and the kernel matrix $K(Q_i, Q_j)$ can be

*—We use the notation "kernel" or "reproducing kernel" for the function $K(P, Q) = K(\psi)$ and reserve the notation "kernel function" for $K(P, Q_i)$.

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