GROUPS ALL PROPER QUOTIENT GROUPS OF WHICH POSSESS LAYER-CHERNIKOV PROPERTIES

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We describe solvable groups all proper quotient groups of which possess layer-Chernikov properties.

In the theory of groups, the groups all proper quotient groups of which (i.e., quotient groups with respect to the nonidentity normal subgroups) possess a certain property or belong to a certain (well-investigated) class of groups have been studied for a long time. Among the first works (in the theory of infinite groups), one should mention the works of Newman [1, 2], where the infinite groups all quotient groups of which are Abelian were investigated. Later, McCarthy [3, 4] and Wilson [5] studied the infinite groups all proper quotient groups of which are finite. This led to the investigation of, on the one hand, groups all proper quotient groups of which are generalized Abelian groups and, on the other hand, groups all proper quotient groups of which satisfy a certain condition of finiteness. For example, Robinson and Wilson [6] described solvable groups all proper quotient groups of which satisfy the Max condition. For solvable groups all quotient groups of which satisfy the dual Min condition, the description was obtained in [7, 8]. Robinson and Zhang [9] studied solvable groups all proper quotient groups of which are finite over the center or have a commutant, i.e., are well-investigated subclasses of the class of FC-groups. Another well-investigated subclass of the class of FC-groups is the class of layer-finite groups (groups in which the set of elements of each order is finite). On the other hand, the class of layer-finite groups is a subclass of the class of layer-Chernikov groups.

The layer-finite groups were introduced by Chernikov in [10, 11]. These groups were completely investigated by Chernikov in [10, 12]. This class remains one of the most completely investigated subclasses of the class of FC-groups. Polovitskii [13, 14] studied periodic groups whose subgroup $G(n)$ is a Chernikov group for any $n \in \mathbb{Z}$. These groups are now known as layer-Chernikov groups. Some problems of the construction of layer-Chernikov groups were studied by Robinson [15]. The class of layer-Chernikov groups is also one of the most completely investigated subclasses of the class of CC-groups (groups with Chernikov classes of conjugate elements). At the same time, these groups possess certain properties that essentially distinguish them from the FC-groups. For this reason, the investigation of groups all proper quotient groups of which are layer-Chernikov groups is a natural and interesting problem.

The present paper is devoted to the investigation of these groups.

Let $G$ be a group, let $A$ be its Abelian normal subgroup, and let $H = G/A$. Then $A$ can be regarded as a $ZH$-module if we define the action of $ZH$ on $A$ by the rule

$$a(n_1 \bar{h}_1 + \ldots + n_k \bar{h}_k) = (a^{h_1})^{n_1} (a^{h_2})^{n_2} \ldots (a^{h_k})^{n_k}, \quad a \in A,$$

where $a^{h_i} = h_i^{-1}ah_i$, $\bar{h}_i = h_iA$, $1 \leq i \leq k$.

Let $A$ be a module over a ring $K$. We say that $A$ is a coquasifinite module (see [16, Sec.1.2]) if each nonzero submodule of it has a finite index and the intersection of all nonzero submodules is the zero submodule.

**Lemma 1.** Let $G$ be an almost solvable periodic group all proper quotient groups of which are layer-Chernikov groups. If $G$ is not a layer-Chernikov group, then $G$ contains an infinite normal elementary Abelian $p$-subgroup, where $p$ is a prime number, such that the $F_p(G/A)$-module $A$ is either simple or coquasifinite.
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In what follows, the symbol \( \subset \) denotes imbedding.

**Proof.** Since the group \( G \) is almost solvable, it contains a normal Abelian subgroup \( B \). If \( B \) contains non-identity \( G \)-admissible subgroups \( B_1 \) and \( B_2 \) such that \( B_1 \cap B_2 = \{1\} \), then, by virtue of the Remak theorem (see, e.g., Theorem I.1.2 in [17]), we get \( G \subset G/B_1 \times G/B_2 \). Since the quotient groups \( G/B_1 \) and \( G/B_2 \) are layer-Chernikov groups, \( G \) is also a layer-Chernikov group. Therefore, we can assume that any two nonidentity \( G \)-admissible subgroups of \( B \) have the nonidentity intersection. This means, in particular, that \( B \) is an Abelian \( p \)-subgroup for a certain prime number \( p \). Let \( C = \Omega_1(B) \) be the lower layer of \( B \). If \( C \) is finite, then it follows from the fact that \( G/C \) is a layer-Chernikov group that \( G \) is also a layer-Chernikov group. For this reason, we consider the case where the subgroup \( C \) is infinite. Let

\[
\mathcal{M} = \{ E \mid E \text{ is a nonidentity } G \text{-admissible subgroup of } C \}.
\]

If \( \cap \mathcal{M} \neq \{1\} \), we set \( A = \cap \mathcal{M} \). Then \( A \) does not contain proper nonidentity \( G \)-admissible subgroups, i.e., the \( \mathbb{F}_p(G/A) \)-module \( A \) is simple.

If \( \cap \mathcal{M} = \{1\} \), we set \( A = C \). If \( E \) is a nonidentity \( G \)-admissible subgroup of \( A \), then the quotient group \( G/E \) is a layer-Chernikov group. Hence, its elementary Abelian subgroup \( A/E \) is finite. This means that the \( \mathbb{F}_p(G/A) \)-module \( A \) is coquasifinite. The lemma is proved.

**Lemma 2.** Let \( G \) be an almost solvable periodic group all proper quotient groups of which are layer-Chernikov groups and let \( G \) be not a layer-Chernikov group. Also assume that \( G \) contains a normal infinite elementary Abelian \( p \)-subgroup \( A \), where \( p \) is a prime number, such that the \( \mathbb{F}_p(G/A) \)-module \( A \) is simple. Then \( A = C_G(A) \).

**Proof.** Let \( C_G(A) \neq A \). We set \( C = C_G(A) \). Since the quotient group \( G/A \) is a layer-Chernikov group, \( C/A \) contains a nonidentity \( G \)-admissible subgroup \( K/A \). Then the subgroup \( K \) is finite over the center and, therefore, by virtue of the Schur theorem (see, e.g., Theorem II.1.4 in [17]), we establish that its commutant \( [K, K] \) is finite. Obviously, the subgroup \( [K, K] \) is \( G \)-admissible. Hence, if we now assume that \( [K, K] \) is a nonidentity subgroup, then the fact that \( G/[K, K] \) is a layer-Chernikov group implies that \( G \) is a layer-Chernikov group.

The contradiction obtained above proves that \( [K, K] = \{1\} \), i.e., the subgroup \( K \) is Abelian. The quotient group \( G/A \) is almost solvable and, consequently, it contains a normal solvable subgroup \( H/A \) with finite index. Theorem proved by Zaitsev in [18] and Theorem A in [19] yield the decomposition \( K = A \times B \), where \( B \) is an \( H \)-admissible subgroup. It is clear that \( B \) is finite and, therefore, the index \( |H:C_H(B)| \) is finite. Since the index \( |G:H| \) is also finite, we conclude that the index \( |G:C_G(B)| \) is finite. Thus, the subgroup \( B^G \) is finite. Since \( K \neq A \), we conclude that \( B \) is a nonidentity group, which means that \( B^G \) is also a nonidentity group. Hence, the group \( G \) contains a finite nonidentity normal subgroup. However, as noted above, in this case this group is a layer-Chernikov group. Thus, we again arrive at a contradiction, which proves the equality \( C_G(A) = A \). Lemma 2 is proved.

**Lemma 3.** Let \( G \) be an almost solvable periodic group all proper quotient groups of which are layer-Chernikov groups and let the group \( G \) be not a layer-Chernikov group. Then \( G \) contains an infinite normal elementary Abelian \( p \)-subgroup \( A = C_G(A) \), where \( p \) is a prime number, such that the \( \mathbb{F}_p(G/A) \)-module \( A \) is simple.

**Proof.** It follows from Lemma 1 that \( G \) contains an infinite elementary Abelian \( p \)-subgroup \( A \), where \( p \) is a prime number, such that either the \( \mathbb{F}_pG \)-module \( A \) is coquasifinite or the \( \mathbb{F}_pG \)-module \( A \) is simple. In the second case, Lemma 2 implies the equality \( A = C_G(A) \). Let us show that the first case is impossible.