We consider the problem of approximation of continuous functions by generalized polynomials in the case where the values of the function at the observation points are known with random errors. We construct confidence limits with a given significance level for the true values of the function at any point of its domain of definition.

In the present paper, the functional Banach space $X[0, 1]$ is understood as any Banach space that is algebraically and topologically imbedded into the space $C[0, 1]$ with Chebyshev norm $[1]$. Thus, the elements of the space $X[0, 1]$ are continuous functions. Furthermore, in this space, the value of a function $f(t) \in X[0, 1]$ at a fixed point $t_0 \in [0, 1]$ is defined, which generates a continuous linear functional $\mu_{r_0}(f) = f(t_0)$ in the space $X[0, 1]$. Indeed, since $X[0, 1] \subset C[0, 1]$, the opposite imbedding $X^*[0, 1] \subset C^*[0, 1]$ is true for the adjoint spaces. Hence, if $\mu_{r_0}(f) \in C^*[0, 1]$, then $\mu_{r_0}(f) \in X^*[0, 1]$.

Let $N_m$ be an $m$-dimensional subspace of the Banach space $X = X[0, 1]$. A continuous linear operator $A$ that acts from $X$ into $N_m$ associates any function $f \in X$ with a generalized polynomial

$$U_m(f, t) = \sum_{k=1}^{m} \mu_k(f) \phi_k(t),$$

where $\mu_k$, $k = 1, 2, \ldots, m$, are continuous linear functionals defined on $X$, and $\phi_k(t)$, $k = 1, 2, \ldots, m$, is a certain basis of the subspace $N_m$.

We regard the generalized polynomial $U_m(f, t)$ as an approximation of the function $f(t)$. In what follows, we omit the term "generalized" and call $U_m(f, t)$ the "polynomial." The error of approximation of the function $f(t)$ by the polynomial $U_m(f, t)$ is denoted by

$$R(t) = f(t) - U_m(f, t).$$

In the case where only the values of the approximated function $f(t)$ at the points $\tau_i$, $i = 0, 1, \ldots, N$, are known, the functional $\mu_k(f)$ is a certain linear combination of these values and, hence, of the functionals $\mu_{\tau_i}(f)$:

$$\mu_k(f) = \sum_{i=0}^{N} \gamma_{ik} f(\tau_i) = \sum_{i=0}^{N} \gamma_{ik} \mu_{\tau_i}(f).$$

Assume that, at the points $\tau_0, \tau_1, \ldots, \tau_N$ we know approximate values of the function $f(t)$

$$\tilde{y}_i = f(\tau_i) + \xi_i, \quad i = 0, 1, \ldots, N,$$
instead of its true values. Here, \( \{\xi_i, i = 0, 1, \ldots, N\} \) is a sequence of jointly independent random variables distributed according to the normal law with parameters \( 0 \) and \( \sigma^2 \).

In this case, we associate the function \( f(t) \in X \) with the polynomial

\[
U_m(\tilde{y}, t) = \sum_{k=1}^{m} \left( \sum_{i=0}^{N} \gamma_{ik} \tilde{y}_k \right) \varphi_k(t),
\]

which is constructed on distorted values \( \tilde{y}_i \) of the function \( f(t) \). The error of approximation of the function \( f(t) \) by this polynomial is denoted by

\[
\tilde{R}(t) = f(t) - U_m(\tilde{y}, t).
\]

It is easy to see that the functions \( U_m(\tilde{y}, t) \) and \( \tilde{R}(t) \) are random.

Let \( X_0 \) be the class of functions from the space \( X[0, 1] \) such that

\[
\sup_{f \in X_0} R(t) = \sup_{f \in X_0} |f(t) - U_m(f, t)| = \psi(t) < \infty, \quad t \in [0, 1].
\]

Then, for any function \( f(t) \in X_0 \), we can indicate the exact limits of error at any point \( t \):

\[
|f(t) - U_m(f, t)| \leq \psi(t), \quad 0 \leq t \leq 1.
\]

If \( f(t) \) is approximated by the random polynomial \( U_m(\tilde{y}, t) \), then, at any fixed point \( t \), the function \( \tilde{R}(t) \) is a normally distributed random variable whose value exceeds any number \( C \in R^1 \) with nonzero probability. Therefore, in this case, one can speak only about the construction of a random confidence interval for the unknown value of \( f(t) \) with given significance level \( 2\beta \), i.e., an interval \((\tilde{a}, \tilde{b})\) with random endpoints \( \tilde{a} \) and \( \tilde{b} \) that contains the value of \( f(t) \) with given probability \( 1 - 2\beta \):

\[
\mathbf{p}(f(t) \in (\tilde{a}, \tilde{b})) = 1 - 2\beta.
\]

Thus, the interval \((\tilde{a}, \tilde{b})\) is a random confidence interval for the deterministic (nonrandom) variable \( f(t) \). Let us construct a confidence interval centered at the point \( U_m(\tilde{y}, t) + R(t) \). In the case of symmetric distributions, this interval has the least length among all random confidence intervals \( (U_m(\tilde{y}, t) + R(t) + \alpha_1, U_m(\tilde{y}, t) + R(t) + \alpha_2) \), \( \alpha_2 - \alpha_1 \geq 0 \), with given significance level \( 1 - 2\beta \) constructed for the deterministic variable \( f(t) \). (Note that the length of the random confidence interval \( (U_m(\tilde{y}, t) + R(t) + \alpha_1, U_m(\tilde{y}, t) + R(t) + \alpha_2) \) is equal to \( \alpha_2 - \alpha_1 \) and, hence, it is a deterministic variable.) In other words, it is necessary to find a function \( g(t) \) for which the following relation is true:

\[
\mathbf{p}(U_m(\tilde{y}, t) + R(t) - g(t) \leq f(t) \leq U_m(\tilde{y}, t) + R(t) + g(t)) = 1 - 2\beta, \quad 0 \leq t \leq 1.
\]

For this purpose, we introduce the normalized Laplace function \([2]\)

\[
\Phi_0(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-v^2/2} dv
\]

and the value \( t_\beta \) such that \( \Phi_0(t_\beta) = 0.5 - \beta \).