ON THE INVARIANCE OF A BOUNDARY-VALUE PROBLEM FOR A NONLINEAR EVOLUTION EQUATION

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We obtain an invariance group for one boundary-value problem in the physics of the sea.

In the present paper, we study the boundary-value problem that often arises in the course of investigation of real physical processes occurring in water media (e.g., the propagation of heat waves in the bulk of the ocean, variation in the salinity of sea water caused by rain, undercurrents, influx of river water, etc.) [1]:

\[ \frac{[f(u_x)u_x]}{x} - u_t = 0, \quad 0 < x < \infty, \quad t > 0, \]

\[ u(x, 0) = 0, \quad 0 < x < \infty, \quad \frac{[f(u_x)u_x]}{x} \bigg|_{x=0} = \frac{q}{\lambda}, \]

\[ \lim_{x \to \infty} u(x, 0) = 0, \quad \lim_{x \to \infty} \frac{[f(u_x)u_x]}{x} = 0, \quad t > 0, \]

where \( q \) and \( \lambda \) are constants describing the state of the water medium and \( f(u_x) \) is a function of the gradient.

The invariance of a solution \( U(x, t) \) of the boundary-value problem (if it exists) follows from the invariance of Eq. (1) and its boundary conditions with respect to a certain (the same) transformation group.

We consider problem (1), (2) by using the well-known fact of the existence and uniqueness of its solution [1]. The following statement is true:

**Theorem 1.** The nonlinear evolution equation (1) with an arbitrary function \( f(u_x) \) is invariant with respect to the Lie algebra with the basis operators

\[ X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = \partial_u, \quad X_4 = x\partial_x + 2t\partial_t + u\partial_u. \]

**Proof.** We prove this theorem by using the group-analysis technique [2]. Note that \( \partial_x, \partial_t, \) and \( \partial_u \) denote the derivatives with respect to the corresponding variables. We regard Eq. (1) as a manifold in the space \( R^3 \) of variables \( x, y, \) and \( u.\)

To simplify the calculation, we denote \( u_x = p, \ u_t = q, \ u_x = r, \ u_{xt} = s, \) and \( u_{tt} = l \) and rewrite Eq. (1) as follows:

\[ \frac{[f(p)p]}{x} - q_t = 0. \]  

(1')

We seek an infinitesimal operator of the transformation group admitted by Eq. (1) in the form

\[ X = \xi \partial_x + \eta \partial_t + \zeta \partial_u. \]

(4)
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The coefficients \( \xi = \xi(x, t, u), \eta = \eta(x, t, u), \) and \( \zeta = \zeta(x, t, u) \) are determined as solutions of the determining equation, which follows from the criterion of invariance with respect to the second extension of the operator \( X - \hat{X} \):

\[
\hat{X} \left[ f(p r + f_r - q) \right]\left|_{(1)}\right. = F[\xi_x + p \xi_u - p \xi_x - p^2 \xi_u - Fr(\eta_x + \eta_u)] - \eta_t - Fr \zeta_w
\]

\[
+ p \xi_t + Fr(p \xi_u + \eta + Fr \eta_u) + F \left[ \zeta_{xx} + 2p \xi_{uu} + p^2 \xi_{uu} - p \xi_{xx} - 2p^2 \xi_{uu} \right]
\]

\[
- Fr(\eta_{xx} + 2p \eta_{uu} - p^2 \eta_{uu}) + r(\zeta_u - 2\xi_x - 3p \xi_u - Fr \eta_u)
\]

\[
- 2s(\eta_x + p \eta_u) - p^3 \xi_{uu} = 0,
\]

where \( F = F(p) = f'p + f \).

Since the quantities \( p, q, r, \) and \( s \) may take arbitrary values and the coordinates \( \xi, \eta, \) and \( \zeta \) do not depend on them, Eq. (5) admits a decomposition in the indicated quantities. By equating the coefficient of \( s \) to zero, we obtain the equality \( (f'p + f)(\xi_t + p \xi_u) = 0 \). In the case where \( f(p) \neq \text{const} \) and \( f(p) \neq -c/p, \) which corresponds to the linearity of Eq. (1), the last equality yields

\[
\eta_x = 0, \quad \eta_u = 0.
\]

(6)

In view of this fact, the decomposition in \( r \) yields

\[
\xi_u = 0, \quad \eta_r = 2\xi_x.
\]

(7)

In the general case, the function \( F(p) \) is an arbitrary function not identically equal to zero. Then the subsequent decomposition of Eq. (5) gives the equalities

\[
\xi_t = 0, \quad \xi_r = 0, \quad \xi_{xx} = 0, \quad \xi_{uu} = 0, \quad \xi_u = \xi_x.
\]

(8)

These equalities supplement systems (6) and (7) to a complete system of determining equations for the coefficients \( \xi, \eta, \) and \( \zeta \) of the basis operators of the Lie algebra of Eq. (1). Solving this system, we obtain \( \xi_1 = 1, \xi_2 = 0, \xi_3 = 0, \xi_4 = x, \eta_1 = 0, \eta_2 = 1, \eta_3 = 0, \eta_4 = 2t, \zeta_1 = 0, \zeta_2 = 0, \zeta_3 = 1, \) and \( \zeta_4 = x. \)

This means that the kernel of the Lie algebra of Eq. (1) with an arbitrary function \( f(u_x) \) is represented by operators (3). Theorem 1 is proved.

To establish the invariance of the boundary-value problem (1), (2), we consider the group of transformations with an infinitesimal operator equal to a linear combination of the basis operators (3):

\[
X = \sum_{i=1}^{4} a_i X_i = (a_4 x + a_1 \partial_x + (a_4 t + a_2) \partial_t + (a_4 u + a_3) \partial_u).
\]

We represent the boundary conditions (2) in a generalized form as follows:

\[
u(x, t_0) = \psi(x), \quad x > x_0, \quad \left[ f(u_x)u_x \right]_{x = x} = \varphi(t),
\]

\[
\lim_{x \to \infty} u(x, t) = 0, \quad \lim_{x \to \infty} [f(u_x)u_x]_x = 0, \quad t > t_0.
\]

(9)