EXACT SOLUTION OF ONE BOUNDARY-VALUE PROBLEM

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We study the boundary-value periodic problem
\[ u_{tt} - u_{xx} = F(x, t), \quad u(0, t) = u(\pi, t) = 0, \quad u(x, t + T) = u(x, t), \quad (x, t) \in \mathbb{R}^2. \]

By using the Vejvoda–Shtedry operator, we determine a solution of this problem.

In [1, 2], it was proved that the problem
\[ u_{tt} - u_{xx} = F(x, t), \quad u(0, t) = u(\pi, t) = 0, \quad u(x, t + T) = u(x, t), \quad (x, t) \in \mathbb{R}^2, \]
may have a classical solution at least in three spaces \( A_1, A_2, \) and \( A_3 \) of functions that correspond to the periods
\[ T_1 = \frac{(2p - 1)\pi}{s}, \quad T_2 = \frac{4\pi p}{2s - 1}, \]
\[ T_3 = \frac{2\pi(2p - 1)}{2s - 1}, \quad p \in \mathbb{Z}, \quad s \in \mathbb{N}. \]

We consider the following spaces of functions: \( C \) is a space of functions of two variables \( x \) and \( t \), which are continuous and bounded on \( \mathbb{R}^2 \); \( G \) is a space of functions of two variables, which are continuous and bounded on \( \mathbb{R}^2 \) together with the derivative with respect to \( t \); \( Q_T \) is a space of functions \( T \)-periodic with respect to \( t \) on \( \mathbb{R}^2 \); in addition, we consider a subspace \( A_1^0 \) of the space \( A_1 \):
\[ A_1^0 = \{ F: F(x, t) = F(\pi - x, t) = F(x, t + T_1) = -F(x, t) = -F(x, -t) \}, \]
where \( T_1 = \pi / q, \quad q \in \mathbb{N} \).

For functions \( F \in C \), we consider the operator

\[ (SF)(x, t) = \frac{1}{4} \int_0^\pi d\xi \int_0^\pi \{ F(\xi, t + \xi - \eta) + F(\xi, t - \xi + \eta) \} d\eta \]
\[ + \frac{1}{4} \int_0^\pi d\xi \int_0^\pi \{ F(\xi, t + \xi - \eta) + F(\xi, t - \xi + \eta) \} d\eta \]
The following statements are true:

**Lemma 1.** For any function \( F \in G_1 \cap Q_T \), the function \( u = SF \) satisfies Eq. (1) and condition (3):

\[
(SF)(x, -t) = -(SF)(x, t),
\]

(6)

\[
(SF)(x, t + T_1) = (SF)(x, t),
\]

(7)

\[
(SF)(\pi - x, t) = (SF)(x, t).
\]

(8)

**Lemma 2.** If \( F \in A_1^0 \cap C \), then

\[
\int_{-\pi}^{\pi} d\xi_1 \int_{t-x}^{t+x} F(\xi_1, \tau) d\tau = 0.
\]

The proofs of Lemmas 1 and 2 can be obtained by direct verification.

**Theorem 1.** If \( F \in A_1^0 \cap C \), then \( u = SF \in A_1^0 \).

**Proof.** Since Lemma 2 is true, to prove Theorem 1, it remains to show that \( (SF)(-x, t) = -(SF)(x, t) \).

Indeed, since the operator \( S \) admits the representation

\[
(SF)(x, t) = -\frac{1}{2} \int_{0}^{\pi} d\xi_1 \int_{t-x}^{t+x} F(\xi_1, \tau) d\tau + \frac{1}{4} \int_{0}^{\pi} d\xi_1 \int_{t-x}^{t+x} F(\xi_1, \tau) d\tau,
\]

we have on the basis of Lemma 3 that

\[
(SF)(-x, t) = -\frac{1}{2} \int_{t-x}^{t+x} F(-\xi_1, \tau) d\tau + \frac{1}{4} \int_{t-x}^{t+x} F(-\xi_1, \tau) d\tau
\]

\[
= \frac{1}{2} \int_{0}^{\pi} dy \int_{t-x+y}^{t-x-y} F(-y, \tau) d\tau - \frac{1}{4} \int_{0}^{\pi} dy \int_{t-x+y}^{t-x-y} F(-y, \tau) d\tau
\]

\[
= \frac{1}{2} \int_{0}^{\pi} dy \int_{t-x+y}^{t-x-y} F(y, \tau) d\tau - \frac{1}{4} \int_{0}^{\pi} dy \int_{t-x+y}^{t-x-y} F(y, \tau) d\tau
\]

\[
= \frac{1}{2} \int_{0}^{\pi} dy \int_{t-x+y}^{t-x-y} F(y, \tau) d\tau + \frac{1}{4} \int_{0}^{\pi} dy \int_{t-x+y}^{t-x-y} F(y, \tau) d\tau
\]

\[
= \frac{1}{4} \int_{0}^{\pi} dy \int_{t-x+y}^{t-x-y} F(y, \tau) d\tau + \frac{1}{4} \int_{0}^{\pi} dy \int_{t-x+y}^{t-x-y} F(y, \tau) d\tau
\]

\[
= -(SF)(x, t).
\]