REMARK ON THE LEBESGUE CONSTANT IN THE ROGOSINSKI KERNEL

V. K. Dzyadyk and I. A. Shevchuk

For every \( n \), we compute the Lebesgue constant of Rogosinski kernel with any preassigned accuracy.

Consider the Lebesgue constant \( L_n \) of the Rogosinski kernel, i.e., the number [1, pp. 121–123]

\[
L_n = \frac{1}{2\pi} \sin \frac{\pi}{2n} \int_{-\pi}^{\pi} \frac{\cos nt}{\cos t - \cos (\pi/2n)} \, dt, \quad n \in \mathbb{N}.
\]

Korneichuk proved that

\[
L_n = \frac{2}{\pi} \int_0^\pi \sin t \, dt - r_n,
\]

where

\[
0 < r_n < \frac{2}{\sqrt{3}(2n+1)}.
\]

One of the authors (V. K. Dzyadyk) proved that

\[
0 < r_n < \frac{5}{12} \frac{1}{n^2}.
\]

A simple corollary of the reasoning of [1, pp. 121–123] and the Euler–Maclaurin summation formula is Proposition 1, which allows one to calculate \( r_n \) for all \( n \) with any preassigned accuracy. For example,

\[
r_n = \frac{1}{6n^2}
\]

with accuracy \((4/15)(2n)^{-4}\);

\[
r_n = \frac{1}{6n^2} + \frac{\pi^2 - 6}{360} \frac{1}{n^4}
\]

with accuracy \((8/63)(2n)^{-6}\);

\[
r_n = \frac{1}{6n^2} + \frac{\pi^2 - 6}{360} \frac{1}{n^4} + \frac{\pi^4 - 20\pi^2 + 120}{15120} \frac{1}{n^6}
\]
with accuracy \((2/27)(2n)^8, \ldots, \text{etc.}\)

Computation of \(r_n\) can be reduced to obtaining the approximation error by the trapezium quadrature formula of the integral of the following function:

\[
f(x) := \frac{\sin x}{x},
\]

\(f(0) := 1,\) since

\[
r_n = \frac{2}{\pi} \int_0^\pi f(t)\,dt - \frac{2}{\pi} \sum_{j=0}^{n-1} \frac{1}{j} \left(f\left(\frac{j\pi}{n}\right) + f\left(\frac{j+1}{n}\pi\right)\right) \frac{\pi}{n}
\]

(1)

according to [2, p. 123, (22)].

Therefore, it is natural to use the Euler–Maclaurin summation formula [3, p. 136, (4.8)–(4.10)]: if a function \(g\) has \(2m+2\) continuous derivatives on \([0, n]\), then

\[
\sum_{j=0}^{n} g(j) = \int_0^n g(t)\,dt + g(0) + g(n) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \left(g^{(2k-1)}(n) - g^{(2k-1)}(0)\right) + \frac{n^{2m+2}}{(2m+2)!} g^{(2m+2)}(0n),
\]

(2)

\(m, n = 1, 2, \ldots, 0 < \theta < 1,\) where

\[
B_{2k} = 2(-1)^{k+1}(2k)! \frac{1}{(2\pi)^{2k}} \sum_{r=1}^{n} \frac{1}{r^{2k}}
\]

(3)

are the Bernoulli numbers [3, p. 747, (21.5)–(21.16)].

The equality

\[
f^{(j)}(x) = \frac{1}{x^{j+1}} \int_0^x t^j \cos\left(t + \frac{j\pi}{2}\right)\,dt, \quad j = 0, 1, 2, \ldots,
\]

(4)

can easily be proved by induction, whence

\[
|f^{(2k)}(t)| < \frac{1}{2k+1}, \quad 0 < t < \pi.
\]

In particular, taking (3) into account, we obtain

\[
\left| \frac{B_{2m+2}}{(2m+2)!} f^{(2m+2)}(0) \right| \leq \frac{2}{2^{2m+2}} \frac{1}{2m+3} \to 0, \quad m \to \infty.
\]

(5)

Relations (1), (2), and (5) (in view of the equality \(f^{(2k-1)}(0) = 0,\) which follows from the evenness of the function \(f\)) prove the validity of the first part of the following assertion:

**Proposition 1.** The equality