SOBOLEV PROBLEM IN THE COMPLETE SCALE OF BANACH SPACES

Ya. A. Roitberg and A. V. Sklyarets

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In a bounded domain $G \subset \mathbb{R}^n$, whose boundary is the union of manifolds of different dimensions, we study the Sobolev problem for a properly elliptic expression of order $2m$. The boundary conditions are given by linear differential expressions on manifolds of different dimensions. We study the Sobolev problem in the complete scale of Banach spaces. For this problem, we prove the theorem on a complete set of isomorphisms and indicate its applications.

1. Introduction

The Sobolev problem in the classes of sufficiently smooth functions was studied sufficiently completely (see [1–3] and the bibliography there). In this paper, we study the Sobolev problem in the complete scale of Banach spaces and establish the theorem on the complete set of isomorphism for this problem. We also introduce and study Dirichlet systems on an $i$-dimensional manifold $\Gamma$ lying inside a domain. In the present paper, we essentially use the theorem on complete set of isomorphisms for elliptic boundary-value problems in domains with smooth $(n - 1)$-dimensional boundary.

2. Statement of the Problem

Let $G \subset \mathbb{R}^n$ be a bounded domain and let $\delta G = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_k$ be its boundary, let $\Gamma_0$ be an $(n - 1)$-dimensional manifold without boundary, which is the exterior boundary of $G$, and let $\Gamma_j$ be an $i_j$-dimensional manifold without boundary, where $0 \leq i_j \leq n - 1$ and $i'_j = n - i_j$ be the codimension of $\Gamma_j$, $j = 1, \ldots, k$, $\Gamma_j \in C^\infty$, $j = 0, \ldots, k$. In $G$, we consider the following boundary-value problem:

$$L(x, D)u = f \quad \text{(in $G$; ord $L = 2m$),}$$

$$B_{j,0}(x, D)u |_{\Gamma_0} = \varphi_{j,0}, \quad j = 1, \ldots, m; \quad \text{ord} B_{j,0} = q_{j,0}, \quad \text{(2)}$$

$$B_{r,k}(x, D)u |_{\Gamma_k} = \varphi_{r,k}, \quad r = 1, \ldots, m_k; \quad k = 1, \ldots, k; \quad \text{ord} B_{r,k} = q_{r,k}. \quad \text{(3)}$$

In what follows, we assume that Eq. (1) is correctly elliptic in $\overline{G}$ and the boundary conditions (2) satisfy on $\Gamma_0$ the Lopatinskii condition [6–8], and for each $k$ the expressions $\{B_{r,k}\}$ forms a Dirichlet system on $\Gamma_k$ (see Sec. 6). Here and in what follows, for simplicity, it is assumed that the coefficients of all differential expressions are smooth, we also assume that $q_{r,k} \leq 2m - i'_k$, $k = 0, \ldots, k$, $r = 1, \ldots, m_k$, $m_0 = m$. To exact the setting of the problem, we have to introduce certain functional spaces.

3. Functional Spaces

Let $p, p' \in (1, \infty)$, $1/p + 1/p' = 1$. By $H^{s, p}(G)$, $s \geq 0$, we denote the space of Bessel potentials (Liouville classes) and, by $H^{-s, p'}(G)$, we denote the space dual to $H^{s, p'}(G)$ with respect to the extension $(\cdot, \cdot)$ of a scalar
product in $L_2(G)$. Let $\| \cdot \|_{s,p}$ be a norm in $H^{s,p}(G)$, $s \in \mathbb{R}$. Note that since for $s \geq 0$ the space $H^{s,p}(G)$ is the space of contractions onto the domain $G$ of elements from $H^{s,p}(\mathbb{R}^2)$ with a quotient topology $H^{s,p}(G) = H^{s,p}(\mathbb{R}^n) / H^{s,p}_{CC}(\mathbb{R}^n)$, where

$$H^{s,p}_{CC}(\mathbb{R}^n) = \{ f \in H^{s,p}(\mathbb{R}^n) : \text{supp} f \subset CG = \mathbb{R}^n \setminus G \},$$

and, for $s \geq 0$, there are no functions with supports on $\Gamma_1 \cup \ldots \cup \Gamma_k$, we have

$$H^{s,p}(G) = H^{s,p}(C \cup \Gamma_1 \cup \ldots \cup \Gamma_k), \quad s \in \mathbb{R}.$$

By $B^{s,p}(\Gamma_k)$, $s \in \mathbb{R}$, $k = 0, \ldots, \bar{k}$, we denote Bessel spaces and $\langle \cdot, \Gamma_k \rangle_{s,p}$ denotes a norm in these spaces. The spaces $B^{-s,p}(\Gamma_k)$ and $B^{-s,p}(\Gamma_k)$ are dual with respect to the extension $\langle \cdot, \cdot \rangle_{\Gamma_k}$ of the scalar product in $L_2(\Gamma_k)$.

Let

$$s \in \mathbb{R}, \quad s \neq j + i'_k / p, \quad j = 0, \ldots, 2m - i'_k, \quad k = 0, \ldots, \bar{k}.$$

By $\tilde{H}^{s,p}(G)$, we denote the completion of $C^\omega(\overline{G})$ in the norm

$$\| u \|_{s,p} = \left( \| u \|_{s,p}^p + \sum_{j=1}^{m} \left\langle \langle D_{y_j}^{s-j} u, \Gamma_0 \rangle \right\rangle_{s-j+p}^p, \quad s \neq j + i'_k / p, \quad j = 0, \ldots, 2m - i'_k, \quad k = 0, \ldots, \bar{k} \right)^{1/p}. \quad (4)$$

Here, $D_{y_j} = i \frac{\partial}{\partial y_j}$, $\nu$ is a normal to $\Gamma_0$. $D_\nu^y = D_\nu'^y \ldots D_\nu'^{i'_k}$, $D_j = i \frac{\partial}{\partial y_j}$, $(y_1, \ldots, y_{i'_k})$ is an orthogonal frame to $\Gamma_k$.

For excluded values of $s$, norm (4) and the space $\tilde{H}^{s,p}(G)$ is determined by using complex interpolation.

A similar space is introduced in [4] and studied in detail in [8] (see also [5], Chap. 3, Sec. 6; [6], I, Chap. 2; [7], Chap. II). The closure $S$ of the mapping

$$u \rightarrow Su = \left( u \big|_{\overline{G}}, u \big|_{\Gamma_0}, \ldots, D_j^{2m-1} u \big|_{\Gamma_0}, \left\{ D_j^0 u \big|_{\Gamma_k}, |\alpha| \leq 2m - i'_k, \quad k = 1, \ldots, \bar{k} \right\} \right), \quad u \in C^\omega(\overline{G}),$$

is the isometry between $H^{s,p}(G)$ and the space of direct product

$$F^{s,p} = H^{s,p}(G) \times \prod_{j=1}^{2m} B^{s-j+1-1/p}, \quad (\Gamma_0) \times \prod_{k=1}^{\bar{k}} B^{s-|\alpha|-i'_k / p} (\Gamma_k). \quad (5)$$

In this case, $S \tilde{H}^{s,p} = F^{s,p}$, if $s < 1 / p$. If $s > 1 / p$, then

$$S \tilde{H}^{s,p} = \left( u_0, u_1, \ldots, u_{2m}, \left\{ u_{\alpha k} : |\alpha| \leq 2m - i'_k, \quad k = 1, \ldots, \bar{k} \right\} : u_j = D_j^{s+1} u_0 \big|_{\Gamma_0} \right)$$

for all $j: s - j + 1 - 1 / p > 0$, $u_{\alpha k} = D_\nu^y u_0 \big|_{\Gamma_0}$, if $s - |\alpha| - i'_k / p > 0$, then the other components $Su$ do not depend on $u_0$ (compare with [6–8]).

In view of this, one can identify $u \in \tilde{H}^{s,p}$ with an element $Su \in F^{s,p}$. We write